## Physics 200 marks

**1.** **At the threshold of static equilibrium, the pulley's angular acceleration is zero, so the tensions on both sides are equal, and friction can act in either direction as needed.**

$N = 4(10)(0.80) = 32\,\text{N}$

$|10m_B - 24| \le 0.50(32)$

$0.80\,\text{kg} \le m_B \le 4.00\,\text{kg}$

**Since** $B = 5\,\text{kg}$ **lies outside this range, B descends and A moves up the incline; kinetic friction then acts down the incline on A.**

$f_k = 0.25(32) = 8\,\text{N}$

$T_A - 24 - 8 = 4a$

$50 - T_B = 5a$

**As the pulley rotates, the two tensions differ; since the string doesn't slip, the linear and angular accelerations are related by** $a = R\alpha$**.**

$I = \frac{1}{2}(2)(0.20)^2 = 0.040\,\text{kg m}^2$

$(T_B - T_A)R = I\alpha = I\frac{a}{R}$

$T_B - T_A = \frac{I}{R^2}a$

$\frac{I}{R^2} = \frac{0.040}{0.20^2} = 1\,\text{kg}$

**Adding the three equations gives the acceleration.**

$a = \frac{50-24-8}{4+5+1} = 1.80\,\text{ms}^{-2}$

$T_A = 4(1.80) + 24 + 8 = 39.2\,\text{N}$

$T_B = 50 - 5(1.80) = 41.0\,\text{N}$

**Since the pulley's centre is fixed, the resultant of all forces on it (including its weight) must be balanced by the axle force. Taking rightward and upward as positive,**

$R_x = T_A \cos 37° = 39.2(0.80) = 31.36\,\text{N}$

$R_y = T_A \sin 37° + T_B + 2(10) = 39.2(0.60) + 41.0 + 20 = 84.52\,\text{N}$

$R_{axle} = \sqrt{R_x^2 + R_y^2} = 90.15\,\text{N}, \beta = \tan^{-1}\left(\frac{R_y}{R_x}\right) = 69.64°$

**Verify using energy conservation: after descending** $2\,\text{m}$ **from rest,**

$v^2 = 2as = 2(1.80)(2) = 7.20\,\text{m}^2\text{s}^{-2}$

$K = \frac{1}{2}(4 + 5 + 1)\,v^2 = \frac{1}{2}(10)(7.20) = 36.0\,\text{J}$

**Subtracting the work done by friction from the loss in potential energy gives the same result.**

$\Delta U - W_f = 50(2) - 4(10)(0.60)(2) - 8(2) = 36.0\,\text{J}$

**2.** **Since the vertical acceleration is zero, the normal reaction is reduced by the upward component of the tension.**

$N = mg - P\sin 37° = 100 - 50(0.60) = 70\,\text{N}$

$P\cos 37° = 40\,\text{N}$ **exceeds the maximum static friction** $f_{s,max} = \mu_s N = 35\,\text{N}$**, so the box moves.**

$f_k = 0.25(70) = 17.5\,\text{N}$

$a = \frac{40-17.5}{10} = 2.25\,\text{ms}^{-2}$

**At the threshold of motion, the horizontal component equals the maximum static friction, where the normal force itself depends on the tension.**

$P_{min}\cos 37° = \mu_s(mg - P_{min}\sin 37°)$

$0.80P_{min} = 0.50(100 - 0.60P_{min}) \Rightarrow P_{min} = 45.45\,\text{N}$

3. **When moving together, friction alone provides the horizontal acceleration of the upper block A.**
$f_{s,max} = 0.40(2)(10) = 8\,\text{N} \Rightarrow a_{max} = 4\,\text{ms}^{-2}$
$F_{max} = (2 + 3)(4) = 20\,\text{N}$
**Since** $F = 25\,\text{N} > F_{max}$**, the static-friction relation no longer applies; the blocks slip relative to each other and kinetic friction acts.**
$f_k = 0.30(2)(10) = 6\,\text{N}$
$a_A = \frac{f_k}{m_A} = 3\,\text{ms}^{-2}$
$a_B = \frac{F - f_k}{m_B} = \frac{25-6}{3} = \frac{19}{3}\,\text{ms}^{-2}$
$a_B - a_A = \frac{19}{3} - 3 = \frac{10}{3}\,\text{ms}^{-2}$
**With zero initial relative velocity,** $0.60 = \frac{1}{2}\left(\frac{10}{3}\right)t^2 \Rightarrow t = 0.600\,\text{s}$

4. **Balancing forces perpendicular to the incline gives the normal reaction and hence the friction.**
$N = 4(10)\cos 30° = 20\sqrt{3}\,\text{N}$, $f = \mu N = \frac{20\sqrt{3}}{4\sqrt{3}} = 5\,\text{N}$
**The weight of the hanging mass exceeds the total resisting force down the incline, so it descends.**
$30\,\text{N} > 4(10)\sin 30° + 5 = 25\,\text{N}$
**Treating both bodies as one system eliminates the tension as an internal force.**
$30 - 4(10)\sin 30° - 5 = (4 + 3)a \Rightarrow a = \frac{5}{7} = 0.714\,\text{ms}^{-2}$
**Applying Newton's second law to the hanging mass,** $30 - T = 3a \Rightarrow T = \frac{195}{7} = 27.86\,\text{N}$

5. **Since the impulse from the string is vertical during the collision, horizontal momentum is conserved; kinetic energy is not conserved in the collision.**
$mu = (M + m)v_0 \Rightarrow v_0 = \frac{0.010(600)}{1.00} = 6.0\,\text{ms}^{-1}$
**Setting the tension to zero at the top for the borderline complete circle and using energy conservation gives the minimum speed.**
$v_{0,min}^2 = 5gl = 50 > v_0^2 = 36$**, so a complete circle is not achieved.**
$u_{min} = \frac{M+m}{m}\sqrt{5gl} = 707.11\,\text{ms}^{-1}$
**Let** $\phi$ **be the angle turned from the lowest point. While the string stays taut, energy conservation and the centripetal-force equation apply.**
$v^2 = v_0^2 - 2gl(1 - \cos\phi)$, $T = (M + m)\left(\frac{v^2}{1} + g\cos\phi\right)$
**Setting** $T = 0$ **for the first time,** $\cos\phi = \frac{2gl - v_0^2}{3gl} = -\frac{8}{15} \Rightarrow \phi = 122.23°$
**Tension cannot go negative beyond this point; the body then moves as a projectile.** $v_s^2 = -gl\cos\phi = \frac{16}{3} \Rightarrow v_s = 2.309\,\text{ms}^{-1}$
**At the instant the string goes slack, find the height and the upward velocity component.**
$h_s = l(1 - \cos\phi) = \frac{23}{15}\,\text{m}$, $\sin\phi = \frac{\sqrt{161}}{15}$, $v_{y,s}^2 = v_s^2\sin^2\phi = \frac{2576}{675}\,\text{m}^2\text{s}^{-2}$
**Adding the extra height gained in free flight gives the maximum height.**
$h_{max} = h_s + \frac{v_{y,s}^2}{2g} = \frac{23}{15} + \frac{2576}{13500} = 1.7241\,\text{m}$

6. **A downward pseudo-force appears in the lift's frame, increasing the effective gravitational acceleration.**
$g_{eff} = g + A = 10 + 2 = 12\,\text{ms}^{-2}$
**The heavier mass moves down relative to the lift; adding the two masses' equations gives the relative acceleration.**
$a_{rel} = \frac{m_1 - m_2}{m_1 + m_2} g_{eff} = \frac{1}{5}(12) = 2.40\,\text{ms}^{-2}$
**The tension follows from the** $3\,\text{kg}$ **mass's equation, and the balance reading is the sum of the tensions on both sides of the pulley.**
$T = m_1(g_{eff} - a_{rel}) = 3(12 - 2.4) = 28.8\,\text{N}$, $S = 2T = 57.6\,\text{N}$
**To get the acceleration relative to the ground, the lift's acceleration must be added vectorially to the relative acceleration. Taking upward as positive,**
$a_{ground} = A - a_{rel} = 2 - 2.40 = -0.40\,\text{ms}^{-2}$**, i.e. the** $3\,\text{kg}$ **mass accelerates downward relative to the ground.**

7. **Taking upward as positive, the velocity just before impact is negative and just after is positive.**
$v_i = -\sqrt{2(10)(5)} = -10\,\text{ms}^{-1}$, $v_f = \sqrt{2(10)(3.2)} = 8\,\text{ms}^{-1}$
**The impulse of the net force equals the change in momentum.**
$J_{net} = m(v_f - v_i) = 0.20(8 - (-10)) = 3.60\,\text{Ns}$
**Subtracting the weight's impulse from the floor's contact force gives the net impulse, so the average contact force cannot be found from** $\frac{J_{net}}{\Delta t}$ **alone.**
$(N_{avg} - mg)\Delta t = J_{net} \Rightarrow N_{avg} = \frac{3.60}{0.040} + 2 = 92.0\,\text{N}$
**For a fixed floor, the coefficient of restitution is the ratio of the speed of separation to the speed of approach.**
$e = \frac{8}{10} = 0.80$

8. **Since the fuel-ejection rate is constant, the mass decreases linearly with time.**
$m(t) = 1000 - 10t$, $t_b = \frac{200}{10} = 20\,\text{s}$, $m_f = 800\,\text{kg}$
**The relative momentum flux of the ejected gas provides an upward thrust; from the rocket equation,**
$F_{th} = u_e\beta = 2000(10) = 20000\,\text{N}$, $m\frac{dv}{dt} = F_{th} - mg$
**As the mass decreases, the acceleration increases under the same thrust.**
$a_0 = \frac{F_{th}}{m_0} - g = \frac{20000}{1000} - 10 = 10\,\text{ms}^{-2}$, $a_f = \frac{F_{th}}{m_f} - g = \frac{20000}{800} - 10 = 15\,\text{ms}^{-2}$
**Since the acceleration is not constant, the simple equations of motion do not apply; the equation with varying mass must be integrated.**
$v(t) = u_e \ln\frac{m_0}{m_0 - \beta t} - gt$
**Substituting** $t = t_b = 20\,\text{s}$, $v_b = 2000\ln\frac{1000}{800} - 10(20) = 246.29\,\text{ms}^{-1}$

9. **With no external horizontal impulse, total momentum is conserved.**
$\vec{p} = 2(6\hat{i} + 2\hat{j}) + 1(-3\hat{i} + 4\hat{j}) = 9\hat{i} + 8\hat{j}\,\text{kgms}^{-1}$
**Dividing the momentum by the total mass gives the combined velocity.**
$\vec{v} = 3\hat{i} + \frac{8}{3}\hat{j}\,\text{ms}^{-1}$, $|\vec{v}| = \frac{\sqrt{145}}{3} = 4.014\,\text{ms}^{-1}$
**Compute the kinetic energy before and after the collision.**
$K_i = \frac{1}{2}(2)(6^2 + 2^2) + \frac{1}{2}(1)(3^2 + 4^2) = 52.5\,\text{J}$
$K_f = \frac{1}{2}(3)\left(3^2 + \left(\frac{8}{3}\right)^2\right) = \frac{145}{6}\,\text{J}$
**Although momentum is conserved, kinetic energy is not; the lost energy goes into deformation, heat, and sound.**
$\Delta K = K_i - K_f = \frac{85}{3} = 28.33\,\text{J}$

**10.** **The particles approach each other; the relative speed of approach gives the meeting time, and A's motion gives the position.**

$t_c = \frac{12}{5-(-1)} = 2.0\,\text{s}$, $x_c = 5(2) = 10.0\,\text{m}$

**With no external horizontal impulse, momentum is conserved; in a perfectly elastic collision, the relative speed of separation equals the relative speed of approach.**

$2v_A + 3v_B = 2(5) + 3(-1) = 7$, $v_B - v_A = 5 - (-1) = 6$

**Solving the two equations,** $v_A = -\frac{11}{5} = -2.2\,\text{ms}^{-1}$, $v_B = \frac{19}{5} = 3.8\,\text{ms}^{-1}$**; A's negative velocity means it moves back to the left after the collision.**

**The centre-of-mass velocity is unchanged by an internal collision.**

$v_{CM} = \frac{7}{5} = 1.4\,\text{ms}^{-1}$, $x_{CM,0} = \frac{2(0)+3(12)}{5} = 7.2\,\text{m}$

$x_{CM}(2) = 7.2 + 1.4(2) = 10.0\,\text{m}$**, which matches** $x_c$**.**

**11.** **The second collision is perfectly inelastic; the bullet's mass must be included in the combined mass.**

$0.020w = (1.98 + 0.020)(3.0) \Rightarrow w = 300\,\text{ms}^{-1}$

**In the first collision, total momentum of the bullet and the first block is conserved.**

$0.020u = 1.00(2.0) + 0.020(300) \Rightarrow u = 400\,\text{ms}^{-1}$

**Compute the energy loss using both moving bodies' kinetic energy before and after the first collision.**

$\Delta K_1 = \frac{1}{2}(0.020)(400)^2 - \left[\frac{1}{2}(1.00)(2.0)^2 + \frac{1}{2}(0.020)(300)^2\right] = 698\,\text{J}$

**Compute the energy loss of the second collision separately.**

$\Delta K_2 = \frac{1}{2}(0.020)(300)^2 - \frac{1}{2}(2.00)(3.0)^2 = 891\,\text{J}$

**12.** **Add the contributions of the point masses to the rod's own end-axis moment of inertia.**

$I = \frac{1}{3}(3)(2)^2 + 1(2)^2 + 2(1)^2 = 10\,\text{kg m}^2$

**The radius of gyration uses the total mass, not just the rod's mass.**

$k = \sqrt{\frac{I}{M_{total}}} = \sqrt{\frac{10}{6}} = 1.291\,\text{m}$

**A constant torque gives a constant angular acceleration.**

$\alpha = \frac{\tau}{I} = \frac{5}{10} = 0.50\,\text{rads}^{-2}$, $\omega = \alpha t = 2.0\,\text{rads}^{-1}$

**Find the rotational kinetic energy.**

$K = \frac{1}{2}I\omega^2 = \frac{1}{2}(10)(2.0)^2 = 20\,\text{J}$

**13.** **Use the pure-rolling condition together with the moment of inertia of a solid sphere about its centre.**

$v = R\omega$, $I = \frac{2}{5}mR^2$

**Static friction prevents slipping and does no work at the point of contact, so all the potential energy converts into translational and rotational kinetic energy.**

$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{7}{10}mv^2$

$v = \sqrt{\frac{10gh}{7}} = \sqrt{20} = 4.472\,\text{ms}^{-1}$, $\omega = \frac{v}{R} = 22.36\,\text{rads}^{-1}$

$K_{trans} = \frac{1}{2}mv^2 = 20\,\text{J}$, $K_{rot} = \frac{1}{2}I\omega^2 = 8\,\text{J}$

**14.** **Taking rightward displacement and counter-clockwise rotation as positive, write the moment of inertia about the centre and the normal reaction.**

$I = mk^2 = 4(0.30)^2 = 0.36\,\text{kg m}^2$, $N = mg - P\sin\theta$

**Write the equations for translation of the centre and rotation about the centre; for pure rolling,** $\alpha = -\frac{a}{R}$**.**

$P\cos\theta + f = ma$, $Pr + fR = I\alpha$

**Eliminating** $f$ **gives the acceleration.**

$a = \frac{P\left(\cos\theta - \frac{r}{R}\right)}{m + \frac{I}{R^2}} = \frac{P(\cos\theta - 0.50)}{6.25}$

**Substituting** $\theta = 30°$ **and** $P = 10\,\text{N}$, $a = 0.5856\,\text{ms}^{-2}$

$f = -\frac{Pr}{R} - \frac{Ia}{R^2} = -6.318\,\text{N}$ **(friction acts to the left),** $N = 40 - P\sin 30° = 35\,\text{N}$

**The friction needed to prevent slipping cannot exceed the maximum static friction.**

$0.6318P \le 0.20(40 - 0.50P) \Rightarrow P_{\max} = \frac{8}{0.7318} = 10.93\,\text{N}$

**The direction of the acceleration is set by the difference between** $\cos\theta$ **and** $\frac{r}{R}$**, valid only within the static-friction limit.**

$\cos\theta_c = \frac{r}{R} = 0.50 \Rightarrow \theta_c = 60°$**; for** $\theta < 60°$ **the acceleration is to the right, for** $\theta > 60°$ **to the left.**

**15.** **The vertical component of the normal force balances the weight, and the horizontal component provides the centripetal force.**

$N\cos\theta = mg$, $N\sin\theta = \frac{mv^2}{r}$

**Dividing the two equations eliminates the mass.**

$v = \sqrt{rg\tan\theta} = \sqrt{80(10)\left(\frac{1}{4}\right)} = 14.14\,\text{ms}^{-1}$

**From** $\tan\theta = \frac{1}{4}$, $\cos\theta = \frac{4}{\sqrt{17}}$**, which gives the normal reaction.**

$N = \frac{mg}{\cos\theta} = \frac{8000\sqrt{17}}{4} = 8246\,\text{N}$

**The** $1.50\,\text{m}$ **distance is measured along the slope, not horizontally, so the height difference uses the sine.**

$h = 1.50\sin\theta = \frac{1.50}{\sqrt{17}} = 0.364\,\text{m}$

**16.** **Since the particle has no vertical acceleration, the vertical component of the tension balances the weight.**

$T\cos 60° = mg \Rightarrow T = 10\,\text{N}$

**From the geometry, the radius of the circle is** $r = l\sin 60° = \frac{\sqrt{3}}{2}\,\text{m}$

**The horizontal component of the tension provides the centripetal force.**

$T\sin 60° = m\omega^2 r \Rightarrow \omega^2 = \frac{g}{l\cos 60°} = 20\,\text{rad}^2\text{s}^{-2}$

$v = \omega r = \sqrt{15} = 3.873\,\text{ms}^{-1}$, $P = \frac{2\pi}{\omega} = 1.405\,\text{s}$

**The centripetal acceleration always points toward the centre of the circle.**

$a_c = \omega^2 r = 10\sqrt{3} = 17.32\,\text{ms}^{-2}$

**17.** **With no external torque while pulling in the arms, angular momentum is conserved.**

$I_i\omega_i = I_f\omega_f \Rightarrow \omega_f = \frac{6(2)}{2} = 6\,\text{rads}^{-1}$

**Although angular momentum is conserved, kinetic energy is not; the person's work increases the energy.**

$W = \frac{1}{2}(2)(6)^2 - \frac{1}{2}(6)(2)^2 = 24\,\text{J}$

**Since there is an external torque during braking, angular momentum conservation no longer applies in this phase.**

$\alpha = -\frac{3}{2} = -1.5\,\text{rads}^{-2}$**,** $t = \frac{6}{1.5} = 4.0\,\text{s}$

**The angular velocity decreases uniformly, so the average angular velocity gives the angular displacement.**

$\phi = \frac{6+0}{2}(4) = 12\,\text{rad}$

**The work done by the brake is negative; the dissipated energy equals the kinetic energy at the start of braking.**

$E_{\text{loss}} = \frac{1}{2}(2)(6)^2 = 36\,\text{J}$

**18.** **The mass flux striking the vane per second is obtained from the cross-sectional area of the flow.**

$\dot{m} = \varrho Av = 1000\pi(0.01)^2(20) = 2\pi\,\text{kgs}^{-1}$

**Although the speed is unchanged, the direction of the velocity changes; find the rate of change of momentum.**

$\vec{v}_i = 20\hat{i}\,\text{ms}^{-1}$**,** $\vec{v}_f = 20\cos 120°\hat{i} + 20\sin 120°\hat{j} = -10\hat{i} + 10\sqrt{3}\hat{j}\,\text{ms}^{-1}$

**The rate of change of momentum equals the force of the vane on the water.**

$\vec{F}_{\text{water}} = \dot{m}(\vec{v}_f - \vec{v}_i) = -60\pi\hat{i} + 20\sqrt{3}\pi\hat{j}\,\text{N}$

**By Newton's third law, the force of the water on the vane is opposite.**

$\vec{F}_{\text{plate}} = 60\pi\hat{i} - 20\sqrt{3}\pi\hat{j}\,\text{N}$**,** $|\vec{F}_{\text{plate}}| = 40\sqrt{3}\pi = 217.66\,\text{N}$

**The net force on the stationary vane must be zero; the supporting force balances the water's force.**

$\vec{F}_{\text{support}} = -60\pi\hat{i} + 20\sqrt{3}\pi\hat{j} = (-188.50\hat{i} + 108.83\hat{j})\,\text{N}$

**19.** **Differentiating the position with respect to time gives the velocity, and differentiating again gives the acceleration.**

$\vec{v} = -6\sin 2t\hat{i} + 4\cos 2t\hat{j}$**,** $\vec{a} = -12\cos 2t\hat{i} - 8\sin 2t\hat{j} = -4\vec{r}$

**Apply the cross product of position and momentum for the z-component of angular momentum.**

$L_z = m(xv_y - yv_x) = \frac{1}{2}\left(12\cos^2 2t + 12\sin^2 2t\right) = 6\,\text{kg m}^2\text{s}^{-1}$

**The force always points opposite to the position vector, i.e. toward the origin, so it is central; since the torque is zero, angular momentum is constant.**

$\vec{F} = m\vec{a} = -2\vec{r}$**,** $\vec{\tau} = \vec{r} \times \vec{F} = 0$

**This is an isotropic restoring force with force constant** $2\,\text{Nm}^{-1}$**; taking the potential energy at the origin as zero,**

$U = \frac{1}{2}(2)r^2 = 9\cos^2 2t + 4\sin^2 2t\,\text{J}$

**Adding the kinetic energy to this, the time-dependent terms cancel.**

$K = \frac{1}{2}mv^2 = 9\sin^2 2t + 4\cos^2 2t\,\text{J}$**,** $E = K + U = 13\,\text{J}$

**20.** **Let $X$ be the wedge's rightward coordinate and $s$ the block's displacement down the incline relative to the wedge. With no external horizontal force, horizontal momentum is conserved.**

$(M+m)\dot{X} + m\dot{s}\cos\theta = 0 \Rightarrow \dot{X} = -\frac{m\cos\theta}{M+m}\dot{s}$

**Since the horizontal position of the centre of mass stays unchanged from the initial state,**

$\Delta X = -\frac{ms\cos\theta}{M+m} = -\frac{\sqrt{3}}{4} = -0.4330\,\text{m}$

**Apply energy conservation; the block's total kinetic energy must be computed in the ground frame, not just relative to the wedge.**

$mgs\sin\theta = \frac{1}{2}\left[m - \frac{m^2\cos^2\theta}{M+m}\right]\dot{s}^2$

$\dot{s} = \sqrt{\frac{2mgs\sin\theta}{m - \frac{m^2\cos^2\theta}{M+m}}} = 4.961\,\text{ms}^{-1}$

**From the relative speed, find the wedge's velocity and the two components of the block's velocity.**

$\dot{X} = -\frac{m\cos\theta}{M+m}\dot{s} = -1.074\,\text{ms}^{-1}$

$v_x = \dot{X} + \dot{s}\cos\theta = 3.223\,\text{ms}^{-1}$, $v_y = -\dot{s}\sin\theta = -2.481\,\text{ms}^{-1}$

**Differentiating the energy relation with respect to $s$ gives the constant relative acceleration.**

$\ddot{s} = \frac{g\sin\theta}{1 - \frac{m\cos^2\theta}{M+m}} = \frac{80}{13}\,\text{ms}^{-2}$

**Differentiating the momentum relation again gives the wedge's acceleration.**

$A = -\frac{m\cos\theta}{M+m}\ddot{s} = -\frac{10\sqrt{3}}{13} = -1.332\,\text{ms}^{-2}$

**The net horizontal force on the wedge is the horizontal component of the normal reaction.**

$M|A| = N\sin 30° \Rightarrow N = 7.994\,\text{N}$

## Mathematics — 200 marks

**21.** **Let** $z = x + iy$

**Given equation:** $|x + iy - 3| + |x + iy + 3| = 10$

$\Rightarrow \sqrt{(x-3)^2 + y^2} + \sqrt{(x+3)^2 + y^2} = 10$

$\Rightarrow \sqrt{(x-3)^2 + y^2} = 10 - \sqrt{(x+3)^2 + y^2}$

**Squaring both sides,** $(x-3)^2 + y^2 = 100 - 20\sqrt{(x+3)^2 + y^2} + (x+3)^2 + y^2$

$\Rightarrow x^2 - 6x + 9 = 100 - 20\sqrt{(x+3)^2 + y^2} + x^2 + 6x + 9$

$\Rightarrow -12x - 100 = -20\sqrt{(x+3)^2 + y^2}$

$\Rightarrow 3x + 25 = 5\sqrt{(x+3)^2 + y^2}$

**Squaring again,** $9x^2 + 150x + 625 = 25(x^2 + 6x + 9 + y^2)$

$\Rightarrow 9x^2 + 150x + 625 = 25x^2 + 150x + 225 + 25y^2$

$\Rightarrow 16x^2 + 25y^2 = 400$

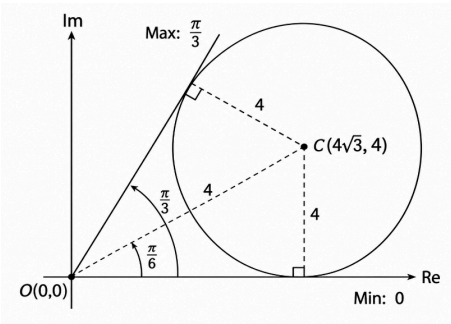
$\Rightarrow \frac{x^2}{25} + \frac{y^2}{16} = 1$

**This is the equation of an ellipse, where** $a^2 = 25 \Rightarrow a = 5$ **and** $b^2 = 16 \Rightarrow b = 4$**.**

**Area of the ellipse** $= \pi ab = \pi(5)(4) = 20\pi$ **square units.**

**22.** **First we express the given complex number** $z = 16\sqrt{3} + 16i$ **in polar form.**
**Here, real part** $x = 16\sqrt{3}$ **and imaginary part** $y = 16$.
**Modulus,** $r = \sqrt{(16\sqrt{3})^2 + (16)^2} = \sqrt{768 + 256} = \sqrt{1024} = 32$
**Since the number lies in the first quadrant,**
**Argument,** $\theta = \tan^{-1}\left(\frac{16}{16\sqrt{3}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$
**So, the general polar form of** $z$**:**
$z = 32\left[\cos\left(2k\pi + \frac{\pi}{6}\right) + i\sin\left(2k\pi + \frac{\pi}{6}\right)\right] = 32e^{i\left(2k\pi + \frac{\pi}{6}\right)}$
$\Rightarrow z = 32e^{i\left(\frac{12k\pi + \pi}{6}\right)}$ **(where** $k \in \mathbb{Z}$**)**
**Applying De Moivre's Theorem, we find the fifth roots of** $z$**:**
$z^{\frac{1}{5}} = \left[32e^{i\left(\frac{12k\pi + \pi}{6}\right)}\right]^{\frac{1}{5}} = (32)^{\frac{1}{5}} e^{i\left(\frac{12k\pi + \pi}{30}\right)} = 2e^{i\left(\frac{12k+1}{30}\right)\pi}$
**To get the** 5 **roots, putting** $k = 0, 1, 2, 3, 4$**:**
**For** $k = 0$**,** $z_1 = 2e^{i\frac{\pi}{30}}$
**For** $k = 1$**,** $z_2 = 2e^{i\frac{13\pi}{30}}$
**For** $k = 2$**,** $z_3 = 2e^{i\frac{25\pi}{30}} = 2e^{i\frac{5\pi}{6}}$
**For** $k = 3$**,** $z_4 = 2e^{i\frac{37\pi}{30}} = 2e^{-i\frac{23\pi}{30}}$
**For** $k = 4$**,** $z_5 = 2e^{i\frac{49\pi}{30}} = 2e^{-i\frac{11\pi}{30}}$
$\therefore$ **The required** 5 **fifth roots are** $2e^{i\frac{\pi}{30}}, 2e^{i\frac{13\pi}{30}}, 2e^{i\frac{5\pi}{6}}, 2e^{-i\frac{23\pi}{30}}, 2e^{-i\frac{11\pi}{30}}$**.**

**23.** **The given inequality represents a circle and its interior.**
**Equation of the circle:** $|z - (4\sqrt{3} + 4i)| = 4$
**Its centre is** $C(4\sqrt{3}, 4)$ **and radius** $r = 4$
**Distance of the centre from the origin** $O(0, 0)$**,**
$OC = \sqrt{(4\sqrt{3})^2 + 4^2} = \sqrt{48 + 16} = \sqrt{64} = 8$
**Since** $OC > r$**, the origin lies completely outside the circle.**
**The two tangents drawn from the origin to the circle mark the maximum and minimum bounds of the argument of** $z$**.**




**Argument of centre** $C$**,** $\theta_c = \tan^{-1}\left(\frac{4}{4\sqrt{3}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$
**Let the tangent drawn from the origin make an angle** $\alpha$ **with the line** $OC$**.**
**From the right-angled triangle formed,**
$\sin\alpha = \frac{r}{OC} = \frac{4}{8} = \frac{1}{2} \Rightarrow \alpha = \frac{\pi}{6}$
$\therefore$ **Maximum argument** $= \theta_c + \alpha = \frac{\pi}{6} + \frac{\pi}{6} = \frac{\pi}{3}$
$\therefore$ **Minimum argument** $= \theta_c - \alpha = \frac{\pi}{6} - \frac{\pi}{6} = 0$
$\therefore$ **Maximum argument of** $z$ **is** $\frac{\pi}{3}$ **and minimum argument is** $0$**.**

**24.** **Let** $z = x + iy \Rightarrow \bar{z} = x - iy$

**Given equation:** $(x + iy)^2 + (x - iy) = 0$

$\Rightarrow x^2 + 2ixy - y^2 + x - iy = 0$

$\Rightarrow (x^2 - y^2 + x) + i(2xy - y) = 0$

**Equating real and imaginary parts from both sides:**

**Real part:** $x^2 - y^2 + x = 0$ **.....(i)**

**Imaginary part:** $2xy - y = 0 \Rightarrow y(2x - 1) = 0$ **.....(ii)**

**From (ii),** $y = 0$ **or** $x = \frac{1}{2}$

**Case 1: When** $y = 0$

**Putting in (i),** $x^2 - 0 + x = 0 \Rightarrow x(x + 1) = 0 \Rightarrow x = 0, -1$

**So, roots:** $z = 0, -1$

**Case 2: When** $x = \frac{1}{2}$

**Putting in (i),** $\left(\frac{1}{2}\right)^2 - y^2 + \frac{1}{2} = 0$

$\Rightarrow \frac{1}{4} + \frac{1}{2} = y^2 \Rightarrow y^2 = \frac{3}{4} \Rightarrow y = \pm\frac{\sqrt{3}}{2}$

**So, roots:** $z = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$\therefore$ **Required solution:** $z = 0, -1, \frac{1}{2} + i\frac{\sqrt{3}}{2}, \frac{1}{2} - i\frac{\sqrt{3}}{2}$**.**

**25.** **Let** $w = \frac{z_1 + z_2}{1 + z_1 z_2}$

**A complex number** $w$ **is real if and only if** $w = \bar{w}$**.**

**We find the conjugate of** $w$**:**

$\bar{w} = \overline{\left(\frac{z_1 + z_2}{1 + z_1 z_2}\right)} = \frac{\bar{z}_1 + \bar{z}_2}{1 + z_1 z_2}$

**Given,** $|z_1| = 1 \Rightarrow |z_1|^2 = 1 \Rightarrow z_1\bar{z}_1 = 1 \Rightarrow \bar{z}_1 = \frac{1}{z_1}$

**Similarly,** $|z_2| = 1 \Rightarrow \bar{z}_2 = \frac{1}{z_2}$

**Putting these values in** $\bar{w}$**:**

$\bar{w} = \frac{\frac{1}{z_1} + \frac{1}{z_2}}{1 + \left(\frac{1}{z_1}\right)\left(\frac{1}{z_2}\right)}$

$\Rightarrow \bar{w} = \frac{\frac{z_1 + z_2}{z_1 z_2}}{\frac{z_1 z_2 + 1}{z_1 z_2}}$

$\Rightarrow \bar{w} = \frac{z_1 + z_2}{1 + z_1 z_2} = w$

**Since** $w = \bar{w}$**, therefore** $\frac{z_1 + z_2}{1 + z_1 z_2}$ **is a real number. (Proved)**

**26.** **Given,** $|z + 2i| \leq 1$

**This represents a circle (or its interior) on the Argand plane with centre** $C_1(0, -2)$ **and radius** $R = 1$**.**

**We need to find the maximum and minimum value of** $|z - 3 + 4i| = |z - (3 - 4i)|$**.**

**Geometrically this is the maximum and minimum distance of the fixed point** $C_2(3, -4)$ **from any point** $z$ **on the circle.**

**If** $d$ **is the distance between the centre** $C_1(0, -2)$ **and the point** $C_2(3, -4)$**,**

$d = \sqrt{(3 - 0)^2 + (-4 - (-2))^2} = \sqrt{9 + 4} = \sqrt{13}$

**We know, for an external point relative to a circle:**

**Maximum distance** $= d + R = \sqrt{13} + 1$

**Minimum distance** $= |d - R| = \sqrt{13} - 1$

$\therefore$ **Maximum value is** $\sqrt{13} + 1$ **and minimum value is** $\sqrt{13} - 1$**.**

**27.** **Given,** $x = -1 + i\sqrt{2}$

$\Rightarrow x + 1 = i\sqrt{2}$

$\Rightarrow (x + 1)^2 = (i\sqrt{2})^2$

$\Rightarrow x^2 + 2x + 1 = -2$

$\Rightarrow x^2 + 2x + 3 = 0$ **.....(i)**

**Now we arrange the given expression** $x^4 + 4x^3 + 6x^2 + 4x + 9$ **in terms of** $(x^2 + 2x + 3)$**:**

$= x^2(x^2 + 2x + 3) + 2x(x^2 + 2x + 3) - 1(x^2 + 2x + 3) + 12$

$= x^2(0) + 2x(0) - 1(0) + 12$ **[using equation (i)]**

$= 12$

**Alternative method:**

$x^4 + 4x^3 + 6x^2 + 4x + 9 = (x + 1)^4 + 8$

$= (i\sqrt{2})^4 + 8 = (\sqrt{2})^4 i^4 + 8 = 4(1) + 8 = 12$**.**

**28.** **Let** $z = x + iy$

$\frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1} = \frac{(x-1)+iy}{(x+1)+iy}$

$= \frac{[(x-1)+iy][(x+1)-iy]}{(x+1)^2+y^2}$

$= \frac{(x^2-1)-i(x-1)y+i(x+1)y-i^2y^2}{(x+1)^2+y^2}$

$= \frac{(x^2+y^2-1)+i(2y)}{(x+1)^2+y^2}$

**Given,** $\text{Arg}\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$

$\therefore \tan^{-1}\left(\frac{\frac{2y}{(x+1)^2+y^2}}{\frac{x^2+y^2-1}{(x+1)^2+y^2}}\right) = \frac{\pi}{4}$

$\Rightarrow \frac{2y}{x^2+y^2-1} = \tan\left(\frac{\pi}{4}\right) = 1$

$\Rightarrow 2y = x^2 + y^2 - 1$

$\Rightarrow x^2 + y^2 - 2y - 1 = 0$

**This is the equation of a circle, with centre** $(0, 1)$ **and radius** $\sqrt{0^2 + (-1)^2 - (-1)} = \sqrt{2}$

$\therefore$ **The equation of the locus is** $x^2 + y^2 - 2y - 1 = 0$ **and it is a circle.**

**29.** **Let,** $P(x) = (x^2 - 3x + 1)(x^2 + cx + 1)$

**On expanding, we get,** $P(x) = x^4 + (c - 3)x^3 + (2 - 3c)x^2 + (c - 3)x + 1$

**Therefore,** $a = c - 3, \quad b = 2 - 3c$

**For** $P(x) = 0$ **to have exactly three distinct real roots, the two roots of** $x^2 + cx + 1 = 0$ **must be equal.**

$c^2 - 4 = 0 \; c = \pm 2$

**When** $c = 2$**,** $a = -1, \quad b = -4$ **and** $x^2 + 2x + 1 = (x + 1)^2$

**Hence, the repeated root is** $-1$**.**

**When** $c = -2$**,** $a = -5, \quad b = 8$ **and** $x^2 - 2x + 1 = (x - 1)^2$

**Hence, the repeated root is** $1$**.**

**Therefore,** $\boxed{(a, b) = (-1, -4), \text{ repeated root } -1}$

**or,** $\boxed{(a, b) = (-5, 8), \text{ repeated root } 1}$

**30.** **Given,** $z = \cos\theta + i\sin\theta$

**Proof of (i):**

**L.H.S** $= \frac{2}{1+\cos\theta+i\sin\theta}$

**Applying the trigonometric identities** $1 + \cos\theta = 2\cos^2\frac{\theta}{2}$ **and** $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$**:**

$= \frac{2}{2\cos^2\frac{\theta}{2}+i\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)}$

$= \frac{2}{2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)}$

$= \frac{1}{\cos\frac{\theta}{2}}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)^{-1}$

**By De Moivre's Theorem:**

$= \frac{1}{\cos\frac{\theta}{2}}\left(\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}\right)$

$= \frac{\cos\frac{\theta}{2}}{\cos\frac{\theta}{2}} - i\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = 1 - i\tan\frac{\theta}{2}$ = **R.H.S (Proved)**

**Proof of (ii):**

**From (i) we have,**

$1 + z = 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)$

$\therefore (1+z)^n = 2^n\cos^n\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)^n$

**Applying De Moivre's Theorem:**

$(1+z)^n = 2^n\cos^n\frac{\theta}{2}\left(\cos\frac{n\theta}{2}+i\sin\frac{n\theta}{2}\right)$ **.....(i)**

**Also,** $\bar{z} = \cos\theta - i\sin\theta$

$1 + \bar{z} = 1 + \cos\theta - i\sin\theta = 2\cos^2\frac{\theta}{2} - i\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)$

$\Rightarrow 1 + \bar{z} = 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}\right)$

$\therefore (1+\bar{z})^n = 2^n\cos^n\frac{\theta}{2}\left(\cos\frac{n\theta}{2}-i\sin\frac{n\theta}{2}\right)$ **.....(ii)**

**Now adding equations (i) and (ii):**

$(1+z)^n + (1+\bar{z})^n = 2^n\cos^n\frac{\theta}{2}\left[\cos\frac{n\theta}{2}+i\sin\frac{n\theta}{2}+\cos\frac{n\theta}{2}-i\sin\frac{n\theta}{2}\right]$

$= 2^n\cos^n\frac{\theta}{2}\left[2\cos\frac{n\theta}{2}\right]$

$= 2^{n+1}\cos^n\left(\frac{\theta}{2}\right)\cos\left(\frac{n\theta}{2}\right)$ **(Proved)**

**31.** **Let the common root of the two equations be** $\alpha$**.**

**Then,**

$a\alpha^2 + b\alpha + c = 0$ **.....(i)**

$c\alpha^2 + b\alpha + a = 0$ **.....(ii)**

**Subtracting (ii) from (i):**

$(a-c)\alpha^2 + (c-a) = 0$

$\Rightarrow (a-c)\alpha^2 - (a-c) = 0$

$\Rightarrow (a-c)(\alpha^2-1) = 0$

**Since it is given** $a \neq c$**, we have** $a - c \neq 0$**.**

**So,** $\alpha^2 - 1 = 0 \Rightarrow \alpha^2 = 1 \Rightarrow \alpha = \pm 1$

**Case 1: When** $\alpha = 1$

**Putting** $\alpha = 1$ **in equation (i):** $a(1)^2 + b(1) + c = 0 \Rightarrow a + b + c = 0$

**Case 2: When** $\alpha = -1$

**Putting** $\alpha = -1$ **in equation (i):** $a(-1)^2 + b(-1) + c = 0 \Rightarrow a - b + c = 0$

$\therefore$ **If the equations have a common root, then** $a + b + c = 0$ **or** $a - b + c = 0$**. (Proved)**

**32.** **Let** $y = \frac{x^2+34x-71}{x^2+2x-7}$

$\Rightarrow y(x^2 + 2x - 7) = x^2 + 34x - 71$

$\Rightarrow yx^2 + 2yx - 7y - x^2 - 34x + 71 = 0$

$\Rightarrow (y - 1)x^2 + 2(y - 17)x - (7y - 71) = 0$ **.....(i)**

**Since** $x$ **is always real, considering equation (i) as a quadratic in** $x$**, its discriminant** $D \geq 0$**.**

$\therefore D = b^2 - 4ac \geq 0$

$\Rightarrow \{2(y - 17)\}^2 - 4(y - 1)\{-(7y - 71)\} \geq 0$

$\Rightarrow 4(y - 17)^2 + 4(y - 1)(7y - 71) \geq 0$

$\Rightarrow (y^2 - 34y + 289) + (7y^2 - 78y + 71) \geq 0$

$\Rightarrow 8y^2 - 112y + 360 \geq 0$

$\Rightarrow y^2 - 14y + 45 \geq 0$

$\Rightarrow y^2 - 9y - 5y + 45 \geq 0$

$\Rightarrow y(y - 9) - 5(y - 9) \geq 0$

$\Rightarrow (y - 5)(y - 9) \geq 0$

**By the rule of inequalities, the product of two factors being non-negative means:**

$y \leq 5$ **or** $y \geq 9$

**That is, the value of** $y$ **is either less than or equal to** $5$**, or greater than or equal to** $9$**. So,** $y$ **can never lie between** $5$ **and** $9$**.**

$\therefore$ **The given statement is completely true. (Verified)**

**33.** **Given equation:** $(x - a)(x - b) + (x - b)(x - c) + (x - c)(x - a) = 0$

$\Rightarrow (x^2 - (a + b)x + ab) + (x^2 - (b + c)x + bc) + (x^2 - (c + a)x + ca) = 0$

$\Rightarrow 3x^2 - 2(a + b + c)x + (ab + bc + ca) = 0$

**Discriminant** $D$ **of the equation:**

$D = \{-2(a + b + c)\}^2 - 4(3)(ab + bc + ca)$

$= 4(a + b + c)^2 - 12(ab + bc + ca)$

$= 4\{(a^2 + b^2 + c^2 + 2ab + 2bc + 2ca) - 3(ab + bc + ca)\}$

$= 4(a^2 + b^2 + c^2 - ab - bc - ca)$

$= 2\{(a - b)^2 + (b - c)^2 + (c - a)^2\}$

**Since the sum of squares of real numbers is always non-negative,**

$D \geq 0$

**So the roots of the equation are always real. (Proved)**

**The condition for equal roots is** $D = 0$

$\Rightarrow (a - b)^2 + (b - c)^2 + (c - a)^2 = 0$

**This is possible only when** $a - b = 0, b - c = 0, c - a = 0$**.**

**That is, the roots will be equal when** $a = b = c$**.**

**34.** **Given, the roots of** $x^2 - px + q = 0$ **are** $\alpha, \beta$**.**

$\therefore \alpha + \beta = p$ **and** $\alpha\beta = q$

**Now, the second equation is:** $q^2x^2 - (p^3 - 3pq)x + q = 0$

**Substituting** $p$ **and** $q$ **in terms of** $\alpha$ **and** $\beta$**:**

$(\alpha\beta)^2x^2 - [(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)]x + \alpha\beta = 0$

$\Rightarrow \alpha^2\beta^2x^2 - (\alpha^3 + \beta^3)x + \alpha\beta = 0$

$\Rightarrow \alpha^2\beta^2x^2 - \alpha^3x - \beta^3x + \alpha\beta = 0$

$\Rightarrow \alpha^2x(\beta^2x - \alpha) - \beta(\beta^2x - \alpha) = 0$

$\Rightarrow (\beta^2x - \alpha)(\alpha^2x - \beta) = 0$

**Either,** $\beta^2x - \alpha = 0 \Rightarrow x = \frac{\alpha}{\beta^2}$

**Or,** $\alpha^2x - \beta = 0 \Rightarrow x = \frac{\beta}{\alpha^2}$

$\therefore$ **The roots of the given equation are** $\frac{\alpha}{\beta^2}$ **and** $\frac{\beta}{\alpha^2}$**.**

**35.** **Since the roots of the equation are in geometric progression (GP),**

**let the three roots be** $\frac{a}{k}, a, ak$

**Product of the roots:** $\left(\frac{a}{k}\right)(a)(ak) = -r$

$\Rightarrow a^3 = -r \Rightarrow a = (-r)^{\frac{1}{3}}$

**Since** $a$ **is a root of the equation, it satisfies it.**

$a^3 + 3pa^2 + 3qa + r = 0$

$\Rightarrow -r + 3pa^2 + 3qa + r = 0$

$\Rightarrow 3pa^2 + 3qa = 0 \Rightarrow 3a(pa + q) = 0$

**Since** $a \neq 0$ **(otherwise** $r = 0$ **and the roots would be** $0$**, which is not a valid GP),**

**we get** $pa + q = 0 \Rightarrow a = -\frac{q}{p}$

**Now putting** $a = -\frac{q}{p}$ **in** $a^3 = -r$**:**

$\left(-\frac{q}{p}\right)^3 = -r$

$\Rightarrow -\frac{q^3}{p^3} = -r$

$\therefore r = \frac{q^3}{p^3}$**.**

**Explanation:** Corrected the derivation path for clarity and consistency: solved directly for $a = -\frac{q}{p}$ from $3a(pa + q) = 0$ (with $a \neq 0$) and substituted into $a^3 = -r$, which gives the same result $r = \frac{q^3}{p^3}$ as the original working.

**36.** **Let the roots of the equation be** $\alpha$ **and** $\alpha^n$**.**

**Sum of roots:** $\alpha + \alpha^n = -\frac{b}{a}$ **.....(i)**

**Product of roots:** $\alpha \cdot \alpha^n = \frac{c}{a}$

$\Rightarrow \alpha^{n+1} = \frac{c}{a}$

$\Rightarrow \alpha = \left(\frac{c}{a}\right)^{\frac{1}{n+1}}$ **.....(ii)**

**Putting the value of** $\alpha$ **from (ii) into (i):**

$\left(\frac{c}{a}\right)^{\frac{1}{n+1}} + \left(\frac{c}{a}\right)^{\frac{n}{n+1}} = -\frac{b}{a}$

**Multiplying both sides by** $a$**:**

$a\left(\frac{c^{\frac{1}{n+1}}}{a^{\frac{1}{n+1}}}\right) + a\left(\frac{c^{\frac{n}{n+1}}}{a^{\frac{n}{n+1}}}\right) = -b$

$\Rightarrow a^{1-\frac{1}{n+1}}c^{\frac{1}{n+1}} + a^{1-\frac{n}{n+1}}c^{\frac{n}{n+1}} = -b$

$\Rightarrow a^{\frac{n}{n+1}}c^{\frac{1}{n+1}} + a^{\frac{1}{n+1}}c^{\frac{n}{n+1}} = -b$

$\therefore (a^nc)^{\frac{1}{n+1}} + (ac^n)^{\frac{1}{n+1}} = -b$**.**

**37.** **1st equation:** $x^2 - 2cx + ab = 0$

**Since its roots are real and unequal, its discriminant** $D_1 > 0$**.**

$D_1 = (-2c)^2 - 4(1)(ab) = 4c^2 - 4ab$

$\therefore 4c^2 - 4ab > 0 \Rightarrow c^2 > ab$ **.....(i)**

**2nd equation:** $x^2 - 2(a + b)x + (a^2 + b^2 + 2c^2) = 0$

**We find its discriminant** $D_2$**:**

$D_2 = \{-2(a + b)\}^2 - 4(1)(a^2 + b^2 + 2c^2)$

$= 4(a + b)^2 - 4(a^2 + b^2 + 2c^2)$

$= 4(a^2 + 2ab + b^2 - a^2 - b^2 - 2c^2)$

$= 4(2ab - 2c^2) = -8(c^2 - ab)$

**From (i), we know** $c^2 > ab \Rightarrow c^2 - ab > 0$

**Since** $(c^2 - ab)$ **is a positive quantity,** $-8(c^2 - ab)$ **must be negative.**

**That is,** $D_2 < 0$

**Since the discriminant of the 2nd equation is negative, its roots are complex (imaginary).**

**(Proved)**

**38.** **Given equation** $x^3 - 2x^2 + 3x - 4 = 0$

**From the relations between the roots of the equation:**

**Sum of the roots,** $\sum \alpha = \alpha + \beta + \gamma = -(-2) = 2$

**Sum of products of roots taken two at a time,** $\sum \alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = 3$

**Product of the roots,** $\alpha\beta\gamma = -(-4) = 4$

**Part 1: Finding** $\sum \alpha^2\beta$

$\Rightarrow \sum \alpha^2\beta = (\sum \alpha)(\sum \alpha\beta) - 3\alpha\beta\gamma$

**Substituting values:**

$\sum \alpha^2\beta = (2)(3) - 3(4) = 6 - 12 = -6$

**Part 2: Finding** $\sum \alpha^3\beta$

**First we find** $\sum \alpha^2$**:** $\sum \alpha^2 = (\sum \alpha)^2 - 2\sum \alpha\beta = (2)^2 - 2(3) = 4 - 6 = -2$

$\Rightarrow \sum \alpha^3\beta = (\sum \alpha^2)(\sum \alpha\beta) - \alpha\beta\gamma(\sum \alpha)$

**Substituting values:**

$\sum \alpha^3\beta = (-2)(3) - (4)(2) = -6 - 8 = -14$

$\therefore$ **The required values are** $-6$ **and** $-14$ **respectively.**

**39.** **Since the three roots of the cubic equation are in arithmetic progression (A.P), let the three roots be:** $a - d, a, a + d$

**Sum of the roots:**

$(a - d) + a + (a + d) = -\frac{-48}{32}$

$\Rightarrow 3a = \frac{48}{32} = \frac{3}{2}$

$\Rightarrow a = \frac{1}{2}$

**Since** $a = \frac{1}{2}$ **is a root of the equation, it satisfies the given equation.**

**Putting** $\frac{1}{2}$ **in place of** $x$**:**

$32\left(\frac{1}{2}\right)^3 - 48\left(\frac{1}{2}\right)^2 + 22\left(\frac{1}{2}\right) - k = 0$

$\Rightarrow 32\left(\frac{1}{8}\right) - 48\left(\frac{1}{4}\right) + 11 - k = 0$

$\Rightarrow 4 - 12 + 11 - k = 0$

$\Rightarrow 3 - k = 0 \Rightarrow k = 3$

**Now, sum of products of roots taken two at a time:**

$(a - d)a + a(a + d) + (a - d)(a + d) = \frac{22}{32}$

$\Rightarrow a^2 - ad + a^2 + ad + a^2 - d^2 = \frac{11}{16}$

$\Rightarrow 3a^2 - d^2 = \frac{11}{16}$

**Putting** $a = \frac{1}{2}$**:**

$3\left(\frac{1}{2}\right)^2 - d^2 = \frac{11}{16}$

$\Rightarrow \frac{3}{4} - d^2 = \frac{11}{16}$

$\Rightarrow d^2 = \frac{3}{4} - \frac{11}{16} = \frac{12-11}{16} = \frac{1}{16}$

$\Rightarrow d = \pm\frac{1}{4}$

**Whether** $d$ **is positive or negative, the set of roots is the same.**

**For** $d = \frac{1}{4}$**, the roots are:** $\frac{1}{2} - \frac{1}{4}, \frac{1}{2}, \frac{1}{2} + \frac{1}{4}$

**That is,** $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$

$\therefore$ **The value of** $k$ **is** $3$ **and the required roots are:** $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$**.**

**40.** **Let the common root of the two equations be $\alpha$.**
**So $\alpha$ satisfies both equations:**
$\alpha^2 + p\alpha + q = 0$ **.....(i)**
$\alpha^2 + q\alpha + p = 0$ **.....(ii)**
**Subtracting (ii) from (i):**
$(p - q)\alpha + (q - p) = 0$
$\Rightarrow (p - q)\alpha - (p - q) = 0$
$\Rightarrow (p - q)(\alpha - 1) = 0$
**Since it is given** $p \neq q$**, we have** $p - q \neq 0$
$\therefore \alpha - 1 = 0 \Rightarrow \alpha = 1$
**Now putting** $\alpha = 1$ **in equation (i):**
$1^2 + p(1) + q = 0$
$\Rightarrow p + q + 1 = 0$ **(Proved)**

**Now, we find the new equation formed by the other roots.**
**Product of roots of the 1st equation** $x^2 + px + q = 0$ **is** $q$
**If one root is the common root** $\alpha = 1$**, let the other root be** $\beta_1$
$\therefore 1 \cdot \beta_1 = q \Rightarrow \beta_1 = q$
**Product of roots of the 2nd equation** $x^2 + qx + p = 0$ **is** $p$
**If one root is the common root** $\alpha = 1$**, let the other root be** $\beta_2$
$\therefore 1 \cdot \beta_2 = p \Rightarrow \beta_2 = p$
**So the roots of the new equation are** $p$ **and** $q$
**Sum of roots** $= p + q$
**Product of roots** $= pq$
**We have proved,** $p + q + 1 = 0 \Rightarrow p + q = -1$
**So, the required equation:**
$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$
$\Rightarrow x^2 - (-1)x + pq = 0$
$\Rightarrow x^2 + x + pq = 0$**.**

## Chemistry **200 marks**

**41.** **Reduction half-reaction:** $MnO_4^- + 8H^+ + 5e^- \rightarrow Mn^{2+} + 4H_2O$ **... (1)**
**Oxidation half-reaction:** $C_2O_4^{2-} \rightarrow 2CO_2 + 2e^-$ **... (2)**
**To balance electrons, multiply equation (1) by** $2$ **and equation (2) by** $5$**, then add:**
**Balanced ionic equation:** $2MnO_4^- + 5C_2O_4^{2-} + 16H^+ \rightarrow 2Mn^{2+} + 10CO_2 + 8H_2O$

**42.** **For a dibasic acid** $H_2A$**, the reaction is:** $H_2A + 2NaOH \rightarrow Na_2A + 2H_2O$
**Moles of** $NaOH$ **used** $= 0.100\,\text{L} \times 0.1\,\text{M} = 0.01\,\text{mol}$
**Moles of** $H_2A$ **required** $= \frac{0.01}{2} = 0.005\,\text{mol}$
**Molecular mass** $M = \frac{\text{mass}}{\text{moles}} = \frac{0.63}{0.005} = 126\,\text{gmol}^{-1}$

**43.** **Given:** $V_1 = 450\,\text{mL}$**,** $T_1 = 27 + 273 = 300\,\text{K}$**,** $P_1 = 750\,\text{mmHg}$
**At STP:** $P_2 = 760\,\text{mmHg}$**,** $T_2 = 273\,\text{K}$
**Formula:** $\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}$
$V_2 = \frac{P_1V_1T_2}{P_2T_1} = \frac{750 \times 450 \times 273}{760 \times 300} = 404.11\,\text{mL}$

**44.** **Moles of** $CO_2$ **gas in** $1.12\,\text{L} = \frac{1.12}{22.4} = 0.05\,\text{mol}$
**Number of molecules** $= 0.05 \times 6.022 \times 10^{23} = 3.011 \times 10^{22}$ **molecules**

45. **The pH range over which phenolphthalein changes colour is** $8.3 - 10.0$**. The equivalence point of a strong acid (**$HCl$**) and a weak base (**$NH_4OH$**) titration lies in the acidic region (pH** $4 - 6$**). Phenolphthalein only changes colour in the alkaline region. Therefore, it remains colourless well before the equivalence point is reached and cannot indicate the correct end point.**

46. **Reaction:** $2KClO_3 \rightarrow 2KCl + 3O_2$
**Moles of** $KClO_3 = \frac{12.25}{122.6} = 0.1\,\text{mol}$
**Moles of** $O_2$ **produced** $= 0.1 \times \frac{3}{2} = 0.15\,\text{mol}$
**Applying** $PV = nRT$**:** $V = \frac{0.15 \times 0.0821 \times 298.15}{1} = 3.67\,\text{L}$

47. **Let the volume be** $V$ L**. Moles of** $H^+ = 0.1V$**, moles of** $OH^- = 2 \times 0.1V = 0.2V$
**Excess moles of** $OH^- = 0.2V - 0.1V = 0.1V$ mol
**The mixture is basic in nature.**
**Total volume** $= 2V$ L
**Concentration of excess** $OH^- = \frac{0.1V}{2V} = 0.05\,\text{M}$

48. **Mole ratio:** $C = \frac{40}{12} = 3.33$**,** $H = \frac{6.67}{1} = 6.67$**,** $O = \frac{53.33}{16} = 3.33$
**Simplest ratio:** $C = 1, H = 2, O = 1 \Rightarrow$ **empirical formula** $= CH_2O$ **(empirical formula mass** $= 30$ **)**
**Molecular mass** $= 2 \times$ vapor density $= 2 \times 30 = 60$
$n = \frac{60}{30} = 2$
**Molecular formula** $= C_2H_4O_2$

49. $1\,\text{mL}$ **of** $H_2O_2$ **solution gives** $20\,\text{mL}$ **of** $O_2$ **at STP.**
$2H_2O_2 \rightarrow 2H_2O + O_2$
$22400\,\text{mL}$ **of** $O_2$ **is obtained from** $68\,\text{g}$ **of** $H_2O_2$**.**
$20\,\text{mL}$ **of** $O_2$ **is obtained from** $\frac{68 \times 20}{22400} = 0.06071\,\text{g}$ **of** $H_2O_2$**.**
**Percentage strength (w/v)** $= 0.06071 \times 100 = 6.071\%$
**Molarity** $= \frac{60.71}{34} = 1.786\,\text{M}$

**50.** **Let, in** 25 mL **of the solution,** NaOH = a **mmol and** $Na_2CO_3$ = b **mmol.**

**Up to the phenolphthalein endpoint,**

$a + b = 15 \times 0.1 = 1.5$

**From the phenolphthalein endpoint to the methyl orange endpoint,**

$b = 10 \times 0.1 = 1.0$

**Therefore,**

$a = 1.5 - 1.0 = 0.5$ mmol

**In** 250 mL **of the solution,**

$n(Na_2CO_3) = 10$ mmol, $n(NaOH) = 5$ mmol

$m(Na_2CO_3) = 0.010 \times 106 = 1.06$ g

$m(NaOH) = 0.005 \times 40 = 0.20$ g

$m(\text{impurity}) = 2.50 - (1.06 + 0.20) = 1.24$ g

**Therefore,** $\boxed{\%Na_2CO_3 = 42.4\%}$

$\boxed{\%NaOH = 8.0\%}$

$\boxed{\%\text{inert impurity} = 49.6\%}$

**51.** $MnO_2 + 4HCl \rightarrow MnCl_2 + Cl_2 + 2H_2O$
$Cl_2 + 2KI \rightarrow 2KCl + I_2$
$I_2 + 2Na_2S_2O_3 \rightarrow 2NaI + Na_2S_4O_6$
$n(S_2O_3^{2-}) = 0.080 \times 0.200 = 0.0160$ mol
$n(I_2) = n(Cl_2) = n(MnO_2) = \frac{0.0160}{2} = 0.00800$ mol
$m(MnO_2) = 0.00800 \times 87 = 0.696$ g
$\%MnO_2 = \frac{0.696}{1.74} \times 100 = 40.0\%$
$m(Na_2S_2O_3 \cdot 5H_2O) = 0.0160 \times 248 = 3.97$ g

**52.** $2Cu^{2+} + 4I^- \rightarrow 2CuI + I_2$
$I_2 + 2S_2O_3^{2-} \rightarrow 2I^- + S_4O_6^{2-}$
$n(S_2O_3^{2-}) = 0.15 \times 0.040 = 6.0 \times 10^{-3}$ mol
$n(I_2) = \frac{6.0 \times 10^{-3}}{2} = 3.0 \times 10^{-3}$ mol
$n(Cu^{2+}) = 2 \times n(I_2) = 6.0 \times 10^{-3}$ mol
$m(Cu) = 6.0 \times 10^{-3} \times 63.5 = 0.381$ g
$\%Cu = \frac{0.381}{3.2} \times 100 = 11.91\%$

**53.** **To find the concentrated solution's molarity, take** 1 L **of solution, with mass** $1000 \times 1.84 = 1840$ g**.**
$m(H_2SO_4) = 0.98 \times 1840 = 1803.2$ g
$M_{conc} = \frac{1803.2}{98} = 18.4$ M
**From the dilution formula** $M_1V_1 = M_2V_2$**,** $M_{dilute} = \frac{18.4 \times 10.0}{500} = 0.368$ M
**Since** $H_2SO_4$ **is diprotic, neutralization requires** $n(NaOH) = 2 \times n(H_2SO_4)$
$V(NaOH) = \frac{0.368 \times 25.0 \times 2}{0.500} = 36.8$ mL
$n(H^+) = 0.368 \times 25.0 \times 2 = 18.4$ mmol

**54.** $n(H_2SO_4) = 0.100 \times 0.2 = 0.02$ mol $\Rightarrow n(H^+) = 0.04$ mol
$n(KOH) = 0.200 \times 0.15 = 0.03$ mol $\Rightarrow n(OH^-) = 0.03$ mol
**Excess** $H^+ = 0.04 - 0.03 = 0.01$ mol
$[H^+] = \frac{0.01}{1} = 0.01$ M
$pH = -\log(0.01) = 2.0$

**55.** **Mass percentage converts directly to ppm:** 0.13% = 1300 ppm
**Taking** 1 L **of solution,** $m(\text{solution}) = 1000 \times 1.025 = 1025$ g
$m(Mg^{2+}) = 0.0013 \times 1025 = 1.3325$ g
$n(Mg^{2+}) = \frac{1.3325}{24.3} = 0.0548$ mol
$[Mg^{2+}] = 1300$ ppm $= 0.0548$ M

**56.** **Mass of solution,** $m = 500 \times 1.00 = 500$ g $= 0.500$ kg
$[As_2O_3] = \frac{10.0 \text{ mg}}{0.500 \text{ kg}} = 20.0$ ppm
**The mass fraction of** As **in** $As_2O_3$ **is** $\frac{2(75)}{2(75)+3(16)} = \frac{150}{198} = 0.7576$
$[As] = 20.0 \times 0.7576 = 15.15$ ppm
**The total mass of** As **is** $10.0 \times 0.7576 = 7.576$ mg**; this mass stays constant on dilution.**
10.0 ppb = 0.0100 ppm**, i.e.** 0.0100 mg **per** kg **of solution**
**The final solution's mass** $= \frac{7.576}{0.0100} = 757.6$ kg $\approx 757.6$ L **(taking density** $\approx 1$ gmL$^{-1}$**)**

**57.** $Na_2CO_3 + 2HCl \rightarrow 2NaCl + H_2O + CO_2$
$n(Na_2CO_3) = \frac{0.265}{106} = 0.0025$ mol $\Rightarrow n(HCl) = 2(0.0025) = 0.005$ mol
$M(HCl) = \frac{0.005}{48.0 \times 10^{-3}} = 0.1042$ M
$CaCO_3 + 2HCl \rightarrow CaCl_2 + H_2O + CO_2$
$n(HCl)_{25mL} = 0.1042 \times 25.0 \times 10^{-3} = 2.604 \times 10^{-3}$ mol
$n(CaCO_3) = \frac{2.604 \times 10^{-3}}{2} = 1.302 \times 10^{-3}$ mol
$m(CaCO_3) = 1.302 \times 10^{-3} \times 100 = 0.1302$ g
**Purity** $= \frac{0.1302}{0.150} \times 100 = 86.8\%$

**58.** $CaOCl_2 + H_2SO_4 \rightarrow CaSO_4 + H_2O + Cl_2$
$Cl_2 + 2KI \rightarrow 2KCl + I_2$
$I_2 + 2Na_2S_2O_3 \rightarrow 2NaI + Na_2S_4O_6$
$n(I_2) = \frac{0.200 \times 0.0280}{2} = 2.80 \times 10^{-3}$ mol
$n(Cl_2) = n(I_2) = 2.80 \times 10^{-3}$ mol
$m(Cl_2) = 2.80 \times 10^{-3} \times 71 = 0.1988$ g
**Available chlorine** $= \frac{0.1988}{1.00} \times 100 = 19.85\%$
$V(Cl_2)_{STP} = 2.80 \times 10^{-3} \times 22400 = 62.7$ mL

**59.** **On heating, only** $NaHCO_3$ **decomposes:** $2NaHCO_3 \rightarrow Na_2CO_3 + H_2O + CO_2$**; the mass loss per mole of** $NaHCO_3$ **is** $\frac{62}{2} = 31\,g$**.**
**Let** $m(NaHCO_3) = x\,g$ **and** $m(Na_2CO_3) = y\,g$**;** $x + y = 13.70$
$\frac{x}{84} \times 31 = 3.10 \Rightarrow x = 8.40\,g$
$y = 13.70 - 8.40 = 5.30\,g$
**The total** $Na_2CO_3$ **in the residue after heating (original** $Na_2CO_3$ **plus that produced from** $NaHCO_3$**) is** $n = \frac{5.30}{106} + \frac{1}{2}\left(\frac{8.40}{84}\right) = 0.050 + 0.050 = 0.100\,mol$
**In the** $25.0\,mL$ **portion of the** $250\,mL$ **solution,** $n(Na_2CO_3) = 0.100 \times \frac{25.0}{250} = 0.0100\,mol$
**Up to the phenolphthalein endpoint, half the** $Na_2CO_3$ **is neutralized (**$Na_2CO_3 \rightarrow NaHCO_3$**), so** $n(HCl) = n(Na_2CO_3) = 0.0100\,mol$
$V_{phenolphthalein} = \frac{0.0100}{0.200} = 0.0500\,L = 50.0\,mL$
**The same amount of** $NaHCO_3$ **is neutralized to reach the methyl orange endpoint, so the additional volume needed is also** $50.0\,mL$**.**

**60.** $Cr_2O_7^{2-} + 6Fe^{2+} + 14H^+ \rightarrow 2Cr^{3+} + 6Fe^{3+} + 7H_2O$
$n(Cr_2O_7^{2-}) = 0.0200 \times 0.0500 = 1.00 \times 10^{-3}\,mol$
**Required** $Fe^{2+}$**:** $6 \times 10^{-3}\,mol$
**Initially,** $n(Fe^{2+}) = 0.100 \times 0.0800 = 8.00 \times 10^{-3}\,mol$
**Since the available** $Fe^{2+}$ **exceeds the required amount,** $K_2Cr_2O_7$ **is the limiting reactant.**
**Remaining** $Fe^{2+} = (8.00 - 6.00) \times 10^{-3} = 2.00 \times 10^{-3}\,mol$
**For the second reaction,** $MnO_4^- + 5Fe^{2+} + 8H^+ \rightarrow Mn^{2+} + 5Fe^{3+} + 4H_2O$
$n(KMnO_4) = \frac{2.00 \times 10^{-3}}{5} = 4.00 \times 10^{-4}\,mol$
$V(KMnO_4) = \frac{4.00 \times 10^{-4}}{0.0200} = 0.0200\,L = 20.0\,mL$
**Therefore,** $K_2Cr_2O_7$ **is the limiting reactant, and** $V(KMnO_4) = 20.0\,mL$**.**