# Engineering Weekly Exam 03 [Home Practice] EV - Answer Sheet

Complex Numbers + Polynomials and Polynomial Equations + Newtonian Mechanics + Quantitative Chemistry

**Total Marks:** 600 **Time:** 3 Hours

Physics: 200 Mathematics: 200 Chemistry: 200

## Physics 200 marks

**1.** **The component of the horizontal force along the incline acts up the slope, while its perpendicular component adds to the perpendicular component of the weight, increasing the normal reaction.**

$N = mg\cos 30° + F\sin 30° = 8(9.8)(0.866) + 60(0.5) = 97.90\,N$

$F\cos 30° - mg\sin 30° = 60(0.866) - 8(9.8)(0.5) = 12.76\,N$

**The required static friction is** $12.76\,N$**, but the maximum static friction is** $f_{s,max} = 0.25(97.90) = 24.47\,N$**; so the block remains at rest.**

**At the threshold of motion,** $F_{min}\cos 30° - mg\sin 30° = \mu_s(mg\cos 30° + F_{min}\sin 30°)$

$F_{min} = \frac{mg(\sin 30° + \mu_s\cos 30°)}{\cos 30° - \mu_s\sin 30°} = 75.81\,N$

**2.** **Since the mass-loss rate is constant,** $m(t) = 1200 - 15t$

$t_b = \frac{300}{15} = 20\,s$**,** $m_b = 900\,kg$

**From the rocket equation, the acceleration is** $a = \frac{u_e\left(-\frac{dm}{dt}\right)}{m} - g$

$a_0 = 2100\left(\frac{15}{1200}\right) - 9.8 = 16.45\,ms^{-2}$

$a_b = 2100\left(\frac{15}{900}\right) - 9.8 = 25.2\,ms^{-2}$

$v_b = 2100\ln\left(\frac{1200}{900}\right) - 9.8(20) = 408\,ms^{-1}$

**As the mass decreases, the acceleration under the same thrust increases with time.**

**3.** **The moment of inertia of the solid disk pulley is** $I = \frac{1}{2}MR^2$**; the heavier mass will descend.**

$\frac{I}{R^2} = \frac{M}{2} = 1\,kg$

$a = \frac{(m_2 - m_1)g}{m_1 + m_2 + \frac{I}{R^2}} = \frac{4(9.8)}{9} = 4.36\,ms^{-2}$

$T_1 - m_1g = m_1a \Rightarrow T_1 = 28.31\,N$

$m_2g - T_2 = m_2a \Rightarrow T_2 = 32.67\,N$

**From the pulley's rotational equation** $(T_2 - T_1)R = I\frac{a}{R}$**,** $T_2 - T_1 = \frac{I}{R^2}a = 4.36\,N$**, consistent with the values above.**

**4.** **Since the person's vertical acceleration relative to the lift is zero, the floor's reaction is the balance reading.**

$N_{up} = m(g + a_1) = 60(9.8 + 1.5) = 678\,N$

$N_{down} = m(g - a_2) = 60(9.8 - 2.0) = 468\,N$

$N_0 = mg = 588\,N$

**Percentage change** $= \frac{N - N_0}{N_0} \times 100\%$

**The reading increases by** $15.3\%$ **during upward acceleration and decreases by** $20.4\%$ **during downward acceleration.**

**5.** **Find the two components of the initial velocity.**
$u_x = 28(0.80) = 22.4\,\text{ms}^{-1}$, $u_y = 28(0.60) = 16.8\,\text{ms}^{-1}$
**Taking the vertical displacement to the ground below the cliff edge as** $-20\,\text{m}$,
$-20 = u_y t - \frac{1}{2}gt^2 \Rightarrow 20 + 16.8t - 4.9t^2 = 0 \Rightarrow t = 4.36\,\text{s}$
$x = u_x t = 22.4(4.36) = 97.8\,\text{m}$
$v_y = u_y - gt = 16.8 - 9.8(4.36) = -25.97\,\text{ms}^{-1}$
$v = \sqrt{u_x^2 + v_y^2} = \sqrt{22.4^2 + 25.97^2} = 34.3\,\text{ms}^{-1}$
$\tan\phi = \frac{|v_y|}{u_x} = \frac{25.97}{22.4} \Rightarrow \phi = 49.2°$

**6.** **Subtract the friction on the incline and on the horizontal block from the driving force along the incline; for the pulley,** $\frac{I}{R^2} = \frac{M}{2} = 0.5\,\text{kg}$.
$f_1 = 0.10(3)(9.8)(0.80) = 2.35\,\text{N}$, $f_2 = 0.20(2)(9.8) = 3.92\,\text{N}$
$a = \frac{3(9.8)(0.60) - f_1 - f_2}{3+2+0.5} = 2.07\,\text{ms}^{-2}$
$3g\sin 37° - f_1 - T_1 = 3a \Rightarrow T_1 = 9.09\,\text{N}$
$T_2 - f_2 = 2a \Rightarrow T_2 = 8.05\,\text{N}$
$T_1 - T_2 = \frac{I}{R^2}a = 0.5(2.07) = 1.04\,\text{N}$, **consistent with the pulley's rotational equation.**

**7.** **The horizontal component of the normal reaction provides the centripetal force, and the vertical component balances the weight.**
$\tan\theta = \frac{v^2}{Rg} \Rightarrow v = \sqrt{\frac{Rg}{3}} = \sqrt{\frac{90(9.8)}{3}} = 17.15\,\text{ms}^{-1}$
**From** $\tan\theta = \frac{1}{3}$, $\cos\theta = \frac{3}{\sqrt{10}}$
$N\cos\theta = mg \Rightarrow N = \frac{mg}{\cos\theta} = 1.03 \times 10^4\,\text{N}$
$\sin\theta = \frac{1}{\sqrt{10}}$, $h = 1.50\sin\theta = 0.474\,\text{m}$

**8.** **With no external horizontal force, the position of the centre of mass stays unchanged. Let the boat's displacement be** $x$, **taken to the left.**
$180x + 60(x + 3) = 0 \Rightarrow x = -\frac{60(3)}{180+60} = -0.75\,\text{m}$
**The person's displacement relative to the ground** $= 3 - 0.75 = 2.25\,\text{m}$
**The negative sign means the boat moves opposite to the person's direction.**

**9.** **From momentum conservation,** $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \Rightarrow 2(8) + 3(-2) = 2v_1 + 3v_2 = 10$
**From the definition of the coefficient of restitution,** $v_2 - v_1 = e(u_1 - u_2) = 0.50(8 - (-2)) = 5$
**Solving the two equations,** $v_1 = -1\,\text{ms}^{-1}$, $v_2 = 4\,\text{ms}^{-1}$
$J_1 = m_1(v_1 - u_1) = 2(-1 - 8) = -18\,\text{Ns}$, $J_2 = m_2(v_2 - u_2) = 3(4 - (-2)) = 18\,\text{Ns}$
$K_i = \frac{1}{2}(2)(8)^2 + \frac{1}{2}(3)(2)^2 = 70\,\text{J}$, $K_f = \frac{1}{2}(2)(1)^2 + \frac{1}{2}(3)(4)^2 = 25\,\text{J}$
$\Delta K = K_i - K_f = 45\,\text{J}$

**10.** $\vec{a}(t) = \frac{\vec{F}}{m} = 2t\hat{i} + 3\hat{j}$
**Integrating with respect to time gives the velocity.**
$\vec{v}(t) = (2 + t^2)\hat{i} + (-1 + 3t)\hat{j}$
$\vec{v}(3) = 11\hat{i} + 8\hat{j}\,\text{ms}^{-1}$, $|\vec{v}| = \sqrt{185} = 13.60\,\text{ms}^{-1}$
**Integrating the velocity again gives the position.**
$\vec{r}(t) = \left(2t + \frac{t^3}{3}\right)\hat{i} + \left(-t + \frac{3t^2}{2}\right)\hat{j}$
$\vec{r}(3) = 15\hat{i} + 10.5\hat{j}\,\text{m}$
**Since the force depends on time, the simple equations of motion (constant acceleration) cannot be used; integration is required.**

**11.** $\vec{\tau} = \vec{r} \times \vec{F} = 6\hat{i} - 4\hat{j} - 14\hat{k}\ \text{Nm}$

**Only $\tau_z$ acts on the rotation about the z-axis.**

$\alpha = \frac{\tau_z}{I} = \frac{-14}{5} = -2.8\ \text{rads}^{-2}$

$\omega = \omega_0 + \alpha t = 10 - 2.8(2) = 4.4\ \text{rads}^{-1}$

$\phi = \omega_0 t + \frac{1}{2}\alpha t^2 = 20 - 5.6 = 14.4\ \text{rad}$

$W = \tau_z \phi = (-14)(14.4) = -201.6\ \text{J}$

**The negative work means the rotational kinetic energy decreases; the same result follows from $\frac{1}{2}I(\omega^2 - \omega_0^2)$.**

**12.** **Add the moments of inertia of the rod and the point masses about the same axis.**

$I = \frac{1}{3}(3)(2)^2 + 1(0.5)^2 + 1(2.0)^2 = 8.25\ \text{kg m}^2$

$\alpha = \frac{\tau}{I} = \frac{10}{8.25} = 1.21\ \text{rads}^{-2}$

$\omega = \alpha t = 2.42\ \text{rads}^{-1}$

$K = \frac{1}{2}I\omega^2 = 24.2\ \text{J}$

**13.** **The moment of inertia of a thin ring is $I = mR^2$; for pure rolling, $v = R\omega$.**

**From energy conservation,** $mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mv^2 \Rightarrow v = \sqrt{gh} = \sqrt{9.8(2.45)} = 4.90\ \text{ms}^{-1}$

$\omega = \frac{v}{R} = \frac{4.90}{0.25} = 19.6\ \text{rads}^{-1}$

$a = \frac{g \sin 30°}{1 + \frac{I}{mR^2}} = \frac{9.8(0.50)}{2} = 2.45\ \text{ms}^{-2}$

$mg \sin 30° - f = ma \Rightarrow f = 7.35\ \text{N}$

**The friction acts up the incline; it produces the ring's rotation and maintains pure rolling.**

**14.** **With zero angular-impulse from external torques about the axis, angular momentum is conserved; kinetic energy is not conserved.**

$I_d = \frac{1}{2}MR^2 = \frac{1}{2}(4)(0.4)^2 = 0.32\ \text{kg m}^2$

$mvR = (I_d + mR^2)\omega$

$\omega = \frac{0.010(400)(0.4)}{0.32 + 0.010(0.4)^2} = 4.98\ \text{rads}^{-1}$

$K_i = \frac{1}{2}mv^2 = 800\ \text{J}$, $K_f = \frac{1}{2}(I_d + mR^2)\omega^2 = 3.98\ \text{J}$

$K_{loss} = 800 - 3.98 = 796\ \text{J}$

**15.** **Find the mass striking the vane per second from the cross-sectional area of the flow.**

$A = \pi(0.02)^2 = 1.256 \times 10^{-3}\ \text{m}^2$

$\dot{m} = \varrho A v = 1000(1.256 \times 10^{-3})(25) = 31.42\ \text{kgs}^{-1}$

**The force of the water on the vane equals the negative rate of change of momentum of the water.**

$\vec{F}_{\text{water on vane}} = \dot{m}(\vec{v}_{in} - \vec{v}_{out}) = 785\hat{i} - 785\hat{j}\ \text{N}$

$|\vec{F}| = \sqrt{785^2 + 785^2} = 1.11 \times 10^3\ \text{N}$

$\tan\phi = \frac{785}{785} \Rightarrow \phi = 45°$ **below the +x axis.**

**Although the jet's speed is unchanged, the direction of the velocity changes, so the rate of change of momentum is not zero.**

**16.** **The force is proportional to $-6\vec{r}$, so it is central and the torque about the origin is zero.**

$\vec{\tau} = \vec{r} \times \vec{F} = 0$

$L_z = m(xv_y - yv_x) = 2[2(-4) - 1(3)] = -22\,\text{kg m}^2\text{s}^{-1}$

$K = \frac{1}{2}mv^2 = \frac{1}{2}(2)(3^2 + 4^2) = 25\,\text{J}$

$U = \frac{1}{2}(6)r^2 = 3(2^2 + 1^2) = 15\,\text{J}$

$E = K + U = 40\,\text{J}$

**Since the torque is zero, the angular momentum stays constant even though the particle's velocity and position change.**

**17.** **With no external horizontal force, the centre-of-mass velocity stays unchanged throughout.**

$v_{CM} = \frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2} = \frac{2(4)+3(0)}{5} = 1.6\,\text{ms}^{-1}$

**At maximum compression, the relative velocity is zero, so both blocks move at $v_{CM}$.**

$v_1 = v_2 = 1.6\,\text{ms}^{-1}$

**Using the reduced mass $\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{6}{5}\,\text{kg}$, energy conservation gives the compression.**

$\frac{1}{2}\mu(4)^2 = \frac{1}{2}kx^2 \Rightarrow x = \sqrt{\frac{\mu(16)}{200}} = 0.310\,\text{m}$

**18.** **At the minimum height, the normal force at the top is zero; the weight alone then provides the centripetal force.**

$mg = \frac{mv_{top}^2}{R} \Rightarrow v_{top}^2 = gR$

**From energy conservation,** $mg(h - 2R) = \frac{1}{2}mv_{top}^2 \Rightarrow h = 2R + \frac{R}{2} = \frac{5R}{2} = 5.0\,\text{m}$

$v_{bottom}^2 = 2gh = 98\,\text{m}^2\text{s}^{-2}$

$N_{bottom} - mg = \frac{mv_{bottom}^2}{R} = 5mg$

$N_{bottom} = 6mg = 6(0.50)(9.8) = 29.4\,\text{N}$

**19.** **The total momentum was zero before the explosion, so it remains conserved.**

$2(6\hat{i}) + 3(-2\hat{j}) + 5\vec{v}_3 = 0 \Rightarrow \vec{v}_3 = -2.4\hat{i} + 1.2\hat{j}\,\text{ms}^{-1}$

$K_1 = \frac{1}{2}(2)(6)^2 = 36\,\text{J}$, $K_2 = \frac{1}{2}(3)(2)^2 = 6\,\text{J}$

$K_3 = \frac{1}{2}(5)(2.4^2 + 1.2^2) = 18\,\text{J}$

$K_{total} = 36 + 6 + 18 = 60\,\text{J}$

**Although momentum is conserved with no external impulse, the internal energy released in the explosion converts into kinetic energy, so the total kinetic energy increases.**

**20.** **On a flat curve, friction alone provides the centripetal force and cannot exceed the maximum static friction.**

$\frac{mv^2}{R} \le \mu mg \Rightarrow R_{min} = \frac{v^2}{\mu g} = \frac{12^2}{0.30(9.8)} = 49.0\,\text{m}$

**The lean angle is given by** $\tan\phi = \frac{v^2}{Rg}$

**For** $R = 50\,\text{m}$, $\tan\phi = \frac{144}{50(9.8)} = 0.294 \Rightarrow \phi = 16.4°$

**At the minimum radius, friction is at its maximum value; for the** $50\,\text{m}$ **curve, the required lean angle is about** $16.4°$**.**

## Mathematics — 200 marks

**21.** **Given quadratic equation:** $z^2 + 4z + (1 - 12i) = 0$

$\Rightarrow z^2 + 4z + 4 - 3 - 12i = 0$

$\Rightarrow (z + 2)^2 = 3 + 12i$

**Let** $z + 2 = w$**, then the equation becomes** $w^2 = 3 + 12i$

**If its roots are** $w_1$ **and** $w_2$**, then** $w_2 = -w_1$**.**

**So the roots of the given equation are** $z_1 = w_1 - 2$ **and** $z_2 = -w_1 - 2$

**We need to find:** $|z_1|^2 + |z_2|^2$

$|z_1|^2 + |z_2|^2 = |w_1 - 2|^2 + |-w_1 - 2|^2 = |w_1 - 2|^2 + |w_1 + 2|^2$

**Using the parallelogram law for complex numbers** $|a - b|^2 + |a + b|^2 = 2(|a|^2 + |b|^2)$**:**

$|w_1 - 2|^2 + |w_1 + 2|^2 = 2(|w_1|^2 + |2|^2) = 2|w_1|^2 + 8$

**Here,** $w_1^2 = 3 + 12i \Rightarrow |w_1^2| = \sqrt{3^2 + 12^2} = \sqrt{9 + 144} = \sqrt{153} = 3\sqrt{17}$

**That is,** $|w_1|^2 = 3\sqrt{17}$

$\therefore |z_1|^2 + |z_2|^2 = 2(3\sqrt{17}) + 8 = 8 + 6\sqrt{17}$**.**

**22. Given,** $|z + 3| = |z - 3|$

**Let** $z = x + iy$

$\Rightarrow |(x + 3) + iy| = |(x - 3) + iy|$

$\Rightarrow (x + 3)^2 + y^2 = (x - 3)^2 + y^2 \Rightarrow x^2 + 6x + 9 = x^2 - 6x + 9 \Rightarrow 12x = 0 \Rightarrow x = 0$

**That is, z is a purely imaginary number. Let** $z = iy$

**Second condition:** $\arg\left(\frac{iy+4}{iy-2i}\right) = \frac{\pi}{4}$

$\frac{4+iy}{i(y-2)} = (4 + iy) \cdot \frac{1}{i(y-2)} = (4 + iy) \cdot \frac{-i}{y-2}$

$= \frac{-4i-i^2y}{y-2} = \frac{y-4i}{y-2} = \frac{y}{y-2} - i\frac{4}{y-2}$

**Since the argument is** $\frac{\pi}{4}$ **(first quadrant), both the real and imaginary parts must be positive and equal.**

$\therefore \frac{y}{y-2} = \frac{-4}{y-2}$

$\Rightarrow y = -4$

**(Check: for** $y = -4$**, real part** $\frac{-4}{-6} = \frac{2}{3} > 0$ **and imaginary part** $\frac{-4}{-6} = \frac{2}{3} > 0$**, so the condition holds).**

$\therefore z = -4i$

**Required value,** $|z|^2 = 0^2 + (-4)^2 = 16$**.**

**23. Given equation:** $z^2 + 6z + 36 = 0$

**We find the roots of the equation:**

$z = \frac{-6 \pm \sqrt{36-4(1)(36)}}{2} = \frac{-6 \pm \sqrt{36-144}}{2}$

$= \frac{-6 \pm \sqrt{-108}}{2} = \frac{-6 \pm 6\sqrt{3}i}{2} = -3 \pm 3\sqrt{3}i$

**Let** $z_1 = -3 + 3\sqrt{3}i$ **and** $z_2 = -3 - 3\sqrt{3}i$

**Value of the first expression:**

$z_1 + 3\sqrt{3}i = -3 + 3\sqrt{3}i + 3\sqrt{3}i = -3 + 6\sqrt{3}i$

$\therefore |z_1 + 3\sqrt{3}i|^2 = (-3)^2 + (6\sqrt{3})^2 = 9 + 108 = 117$

**Value of the second expression:**

$z_2 + 3\sqrt{3}i = -3 - 3\sqrt{3}i + 3\sqrt{3}i = -3$

$\therefore |z_2 + 3\sqrt{3}i|^2 = (-3)^2 + 0^2 = 9$

**Required value:**

$|z_1 + 3\sqrt{3}i|^2 + |z_2 + 3\sqrt{3}i|^2 = 117 + 9 = 126$**.**

**24.** **Given,** $z = \frac{\sqrt{3}}{2} + \frac{i}{2}$

**We express it in polar or Euler form:**

**Modulus** $r = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$

**Argument** $\theta = \tan^{-1}\left(\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$\therefore z = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = e^{i\frac{\pi}{6}}$

**By De Moivre's Theorem:**

$z^{105} = \left(e^{i\frac{\pi}{6}}\right)^{105} = e^{i\frac{105\pi}{6}} = e^{i\frac{35\pi}{2}}$

**We simplify the angle:**

$\frac{35\pi}{2} = 16\pi + \frac{3\pi}{2}$

**Since** $16\pi$ **is a whole multiple of** $2\pi$**,**

$e^{i\frac{35\pi}{2}} = e^{i\left(16\pi + \frac{3\pi}{2}\right)} = e^{i\frac{3\pi}{2}}$

$e^{i\frac{3\pi}{2}} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = 0 + i(-1) = -i$

**Substituting this value into the given expression:**

$(z^{105} - i)^6 = (-i - i)^6 = (-2i)^6$

$= (-2)^6 \cdot i^6 = 64 \cdot (i^2)^3 = 64 \cdot (-1)^3 = -64.$

**25.** **Let** $z = re^{i\theta}$**. Then** $\bar{z} = re^{-i\theta}$**.**

**Given equation:** $2z^2 + \bar{z} = 0$

$\Rightarrow 2r^2e^{i2\theta} + re^{-i\theta} = 0$

**Since we are looking for nonzero roots,** $r \neq 0$**.**

**Dividing the equation by** $re^{-i\theta}$**:**

$2re^{i3\theta} + 1 = 0 \Rightarrow 2re^{i3\theta} = -1$

**Expressing** $-1$ **in polar form,** $-1 = 1 \cdot e^{i(2k\pi+\pi)}$**.**

$\therefore 2re^{i3\theta} = 1 \cdot e^{i(2k\pi+\pi)}$

**Equating the moduli from both sides:**

$2r = 1 \Rightarrow r = \frac{1}{2}$

**Equating the arguments:**

$3\theta = 2k\pi + \pi \Rightarrow \theta = \frac{(2k+1)\pi}{3}$

**For** $k = 0, 1, 2$**, we get distinct values of** $\theta$**:** $\frac{\pi}{3}, \pi, \frac{5\pi}{3}$**.**

**So, there are a total of** $3$ **nonzero roots, each with modulus** $r = \frac{1}{2}$**.**

**Required sum of squares of moduli:**

$\sum |z|^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}.$

**26.** **Equation of circle** $C_1$**:** $|z| = 4$

**Its centre is** $O_1(0, 0)$ **and radius** $R_1 = 4$

**Equation of circle** $C_2$**:** $|z - (5 + 12i)| = 6$

**Its centre is** $O_2(5, 12)$ **and radius** $R_2 = 6$

**Distance between the centres of the two circles,** $d = O_1O_2$**:**

$d = \sqrt{(5-0)^2 + (12-0)^2} = \sqrt{25 + 144} = \sqrt{169} = 13$

**We know, for any two points lying on two circles:**

**Minimum distance,** $\min|z_1 - z_2| = d - R_1 - R_2$ **(if they don't intersect)**

$= 13 - 4 - 6 = 3$

**Maximum distance,** $\max|z_1 - z_2| = d + R_1 + R_2$

$= 13 + 4 + 6 = 23$

**Required value:**

$\max|z_1 - z_2| + \min|z_1 - z_2| = 23 + 3 = 26.$

**27.** **The two given equations represent two circles.**
**Centre of the 1st circle** $C_1(8, 0)$ **and radius** $R_1 = 5$
**Centre of the 2nd circle** $C_2(2, 8)$ **and radius** $R_2 = 5$
**Distance between the centres,** $d = C_1C_2$**:**
$d = \sqrt{(8-2)^2 + (0-8)^2} = \sqrt{6^2 + (-8)^2} = \sqrt{36+64} = \sqrt{100} = 10$
**Here we see,** $R_1 + R_2 = 5 + 5 = 10 = d$
**So, the two circles touch each other externally. They will have exactly one common point (point of tangency), which represents** $z$**.**
**The point** $z$ **will be the midpoint of** $C_1$ **and** $C_2$**.**
$\therefore z = \frac{8+2}{2} + i\frac{0+8}{2} = 5 + 4i$
**We need to find:** $z^2 - 10z + 45$
$z = 5 + 4i \Rightarrow z - 5 = 4i$
**Squaring both sides:**
$(z-5)^2 = (4i)^2$
$\Rightarrow z^2 - 10z + 25 = -16$
$\Rightarrow z^2 - 10z + 41 = 0$
**Given expression:** $z^2 - 10z + 45 = (z^2 - 10z + 41) + 4 = 0 + 4 = 4$**.**

**28.** **Given equation:** $5\left(\frac{x}{1+2i} + \frac{y}{2-i}\right) = 11 - 2i$
**Multiplying numerator and denominator by the conjugate:**
$5\left(\frac{x(1-2i)}{(1+2i)(1-2i)} + \frac{y(2+i)}{(2-i)(2+i)}\right) = 11 - 2i$
$\Rightarrow 5\left(\frac{x(1-2i)}{1^2+2^2} + \frac{y(2+i)}{2^2+1^2}\right) = 11 - 2i$
$\Rightarrow 5\left(\frac{x-2xi}{5} + \frac{2y+yi}{5}\right) = 11 - 2i$
**Cancelling the multiplication and division by** $5$**:**
$(x + 2y) - i(2x - y) = 11 - 2i$
**Equating real and imaginary parts from both sides:**
$x + 2y = 11$ **.....(i)**
$-(2x - y) = -2 \Rightarrow 2x - y = 2$ **.....(ii)**
**From (ii),** $y = 2x - 2$**. Putting this in (i):**
$x + 2(2x - 2) = 11$
$\Rightarrow x + 4x - 4 = 11 \Rightarrow 5x = 15 \Rightarrow x = 3$
**Putting the value of** $x$**,** $y = 2(3) - 2 = 6 - 2 = 4$
$\therefore x + y = 3 + 4 = 7$**.**

**29.** **The given inequality represents a circle and its interior.**
**Equation of the circle:** $|z - 4| = 2$
**Its centre is** $C(4, 0)$ **and radius** $r = 2$
**The tangent drawn from the origin** $O(0, 0)$ **to the circle marks the maximum bound of the argument of** $z$**.**
**Let the tangent drawn from the origin make an angle** $\theta$ **with the** $x$**-axis, which is the maximum argument of** $z$**.**
**From the right-angled triangle** $OTC$**,** $\angle OTC = 90°$**.**
$\sin\theta = \frac{CT}{OC} = \frac{r}{OC} = \frac{2}{4} = \frac{1}{2}$
$\Rightarrow \theta = \frac{\pi}{6}$
**We find the distance** $OT$ **of the point of tangency** $T$ **(which is the required** $z$**) from the origin, using Pythagoras:**
$OT = \sqrt{OC^2 - CT^2} = \sqrt{4^2 - 2^2} = \sqrt{16 - 4} = \sqrt{12} = 2\sqrt{3}$
**So, the modulus of** $z$ **is** $2\sqrt{3}$ **and the argument is** $\frac{\pi}{6}$**. In polar form:**
$z = 2\sqrt{3}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$
$= 2\sqrt{3}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = 3 + i\sqrt{3}$**.**

**30.** **Given inequality:** $|2z - 4(1 + i)| \leq 6$
**Dividing both sides by** $2$**:**
$|z - 2(1 + i)| \leq 3 \Rightarrow |z - (2 + 2i)| \leq 3$
**This represents a circle (or its interior) on the Argand plane with centre** $C(2, 2)$ **and radius** $R = 3$**.**
**We need to find the minimum value of** $|z + 2 - i| = |z - (-2 + i)|$**.**
**Geometrically this represents the minimum distance from a fixed point** $P(-2, 1)$ **to any point** $z$ **on or inside the circle.**
**Distance** $d$ **between the centre** $C(2, 2)$ **and the point** $P(-2, 1)$**:**
$d = \sqrt{(2 - (-2))^2 + (2 - 1)^2} = \sqrt{4^2 + 1^2} = \sqrt{16 + 1} = \sqrt{17}$
**Since** $d = \sqrt{17} \approx 4.12 > R$**, the point** $P$ **lies outside the circle.**
**Minimum distance from a point outside the circle to the circle** $= d - R$
$= \sqrt{17} - 3$**.**

**31.** **1st equation:** $x^2 - 7x + r = 0$
**With roots** $\alpha, \beta$**,**
$\alpha + \beta = 7$ **.....(i)**
$\alpha\beta = r$ **.....(ii)**
**2nd equation:** $x^2 - 5x + r = 0$
**With roots** $2\alpha$ **and** $\frac{\beta}{2}$**,**
$2\alpha + \frac{\beta}{2} = 5 \Rightarrow 4\alpha + \beta = 10$ **.....(iii)**
$(2\alpha)\left(\frac{\beta}{2}\right) = r \Rightarrow \alpha\beta = r$ **(which agrees with the 2nd condition).**

**Now subtracting (i) from (iii):**
$(4\alpha + \beta) - (\alpha + \beta) = 10 - 7$
$\Rightarrow 3\alpha = 3 \Rightarrow \alpha = 1$

**Putting** $\alpha = 1$ **in equation (i):**
$1 + \beta = 7 \Rightarrow \beta = 6$

**Now putting the values of** $\alpha$ **and** $\beta$ **in equation (ii):**
$r = \alpha\beta = (1)(6) = 6$**.**

**32.** **Given equation:** $\lambda x^2 - (\lambda + 4)x + 4 = 0$
**We factorize the equation:**
$\lambda x^2 - \lambda x - 4x + 4 = 0$
$\Rightarrow \lambda x(x - 1) - 4(x - 1) = 0 \Rightarrow (\lambda x - 4)(x - 1) = 0$
$\therefore x = 1$ **or** $x = \frac{4}{\lambda}$
**We are given the condition,** $\alpha < \beta$ **and** $\frac{1}{\alpha} - \frac{1}{\beta} = \frac{1}{2}$
**Case 1: Let** $\alpha = 1$ **and** $\beta = \frac{4}{\lambda}$ **(where** $1 < \frac{4}{\lambda}$**)**
**By the condition,** $\frac{1}{1} - \frac{1}{\frac{4}{\lambda}} = \frac{1}{2} \Rightarrow 1 - \frac{\lambda}{4} = \frac{1}{2}$
$\Rightarrow \frac{\lambda}{4} = \frac{1}{2} \Rightarrow \lambda = 2$
**(Check: for** $\lambda = 2$**, the roots are** $1, 2$**. Here** $1 < 2$ **satisfies the condition.)**
**Case 2: Let** $\alpha = \frac{4}{\lambda}$ **and** $\beta = 1$ **(where** $\frac{4}{\lambda} < 1$**)**
**By the condition,** $\frac{1}{\frac{4}{\lambda}} - \frac{1}{1} = \frac{1}{2} \Rightarrow \frac{\lambda}{4} - 1 = \frac{1}{2}$
$\Rightarrow \frac{\lambda}{4} = \frac{3}{2} \Rightarrow \lambda = 6$
**(Check: for** $\lambda = 6$**, the roots are** $\frac{2}{3}, 1$**. Here** $\frac{2}{3} < 1$ **satisfies the condition.)**
$\therefore$ **The possible values of** $\lambda$ **are** $2$ **and** $6$**.**
**Their sum** $= 2 + 6 = 8$**.**

**33.** **Given equation:** $x^2 - \sqrt{3}x + 3 = 0$

**We find the roots of the equation:**

$x = \frac{\sqrt{3} \pm \sqrt{(-\sqrt{3})^2 - 4(1)(3)}}{2} = \frac{\sqrt{3} \pm \sqrt{3-12}}{2}$

$= \frac{\sqrt{3} \pm \sqrt{-9}}{2} = \frac{\sqrt{3} \pm 3i}{2} = \sqrt{3}\left(\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right)$

**Notice that,** $\frac{1}{2} \pm i\frac{\sqrt{3}}{2} = \cos\frac{\pi}{3} \pm i\sin\frac{\pi}{3} = e^{\pm i\frac{\pi}{3}}$

$\therefore \alpha = \sqrt{3}e^{i\frac{\pi}{3}}$ **and** $\beta = \sqrt{3}e^{-i\frac{\pi}{3}}$

**Applying De Moivre's Theorem:**

$\alpha^{10} + \beta^{10} = (\sqrt{3})^{10}e^{i\frac{10\pi}{3}} + (\sqrt{3})^{10}e^{-i\frac{10\pi}{3}}$

$= 3^5\left(e^{i\frac{10\pi}{3}} + e^{-i\frac{10\pi}{3}}\right)$

$= 243 \cdot 2\cos\left(\frac{10\pi}{3}\right)$

**Now,** $\cos\left(\frac{10\pi}{3}\right) = \cos\left(\frac{10\pi}{3} - 2\pi\right) = \cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2}$

$\therefore \alpha^{10} + \beta^{10} = 243 \cdot 2\left(-\frac{1}{2}\right) = -243$**.**

**Explanation:** Simplified the angle reduction step directly ($\frac{10\pi}{3} - 2\pi = \frac{4\pi}{3}$) instead of via $4\pi - \frac{2\pi}{3}$; both give the same value $\cos\frac{4\pi}{3} = -\frac{1}{2}$, so the final answer $-243$ is unchanged.

**34.** **Given equation:** $x^3 + 0 \cdot x^2 + px + q = 0$

**From the relations between the roots of the equation:**

$\alpha + \beta + \gamma = 0$

$\alpha\beta + \beta\gamma + \gamma\alpha = p$

$\alpha\beta\gamma = -q$

**According to the question,** $-\alpha = 2 \Rightarrow \alpha = -2$

**and** $\beta\gamma = 2$

**Since** $\alpha = -2$ **is a root of the equation, it satisfies it:**

$(-2)^3 + p(-2) + q = 0 \Rightarrow -8 - 2p + q = 0 \Rightarrow 2p - q = -8$ **.....(i)**

**Also, from the product of the roots:**

$\alpha(\beta\gamma) = -q \Rightarrow (-2)(2) = -q \Rightarrow q = 4$

**Putting** $q = 4$ **in equation (i):**

$2p - 4 = -8 \Rightarrow 2p = -4 \Rightarrow p = -2$

**We need to find:** $p^3 + q^3 - 3\alpha\beta\gamma$

**Here** $p = -2$**,** $q = 4$ **and** $\alpha\beta\gamma = -q = -4$

$\therefore p^3 + q^3 - 3\alpha\beta\gamma = (-2)^3 + (4)^3 - 3(-4)$

$= -8 + 64 + 12 = 68$**.**

**35.** **For the equation to be quadratic,** $k - 2 \neq 0 \Rightarrow k \neq 2$
**Condition for two distinct real roots: discriminant** $D > 0$
$D = (-4k)^2 - 4(k-2)(5k) > 0$
$\Rightarrow 16k^2 - 20k(k-2) > 0$
$\Rightarrow 16k^2 - 20k^2 + 40k > 0$
$\Rightarrow -4k^2 + 40k > 0 \Rightarrow k^2 - 10k < 0 \Rightarrow k(k-10) < 0$
$\therefore 0 < k < 10$ **.....(i)**
**Condition for both roots to be positive:**
**i. Sum of roots** $> 0$
$\Rightarrow \frac{-(-4k)}{k-2} > 0 \Rightarrow \frac{4k}{k-2} > 0$
**By the rule of inequalities,** $k \in (-\infty, 0) \cup (2, \infty)$ **.....(ii)**
**ii. Product of roots** $> 0$
$\Rightarrow \frac{5k}{k-2} > 0$
**By the rule of inequalities,** $k \in (-\infty, 0) \cup (2, \infty)$ **.....(iii)**
**Now we find the intersection of (i), (ii) and (iii):**
**The intersection of** $0 < k < 10$ **and** $k \in (-\infty, 0) \cup (2, \infty)$ **is** $2 < k < 10$**.**
**So,** $k$ **can be any integer strictly between** $2$ **and** $10$**.**
**The integer values are:** $3, 4, 5, 6, 7, 8, 9$
$\therefore$ **A total of** $7$ **integer values of** $k$ **are possible.**

**36.** **Given equation:** $x^2 - 7|x| + 12 = 0$
**We know that, for real numbers,** $x^2 = |x|^2$**.**
**We write the equation as a quadratic in** $|x|$**:**
$|x|^2 - 7|x| + 12 = 0$
**Factorizing:**
$|x|^2 - 3|x| - 4|x| + 12 = 0$
$\Rightarrow |x|(|x| - 3) - 4(|x| - 3) = 0$
$\Rightarrow (|x| - 3)(|x| - 4) = 0$
**Either,** $|x| - 3 = 0 \Rightarrow |x| = 3 \Rightarrow x = \pm 3$
**Or,** $|x| - 4 = 0 \Rightarrow |x| = 4 \Rightarrow x = \pm 4$
**So the equation has** $4$ **real roots:** $3, -3, 4, -4$
**Required sum of squares of the roots:**
$S = (3)^2 + (-3)^2 + (4)^2 + (-4)^2$
$= 9 + 9 + 16 + 16 = 50$**.**

**37.** **Let** $A = x^2 - x$ **and** $B = 3x - 15$
**Notice that,** $A + B = (x^2 - x) + (3x - 15) = x^2 + 2x - 15$
**The given equation is of the form** $|A + B| = |A| + |B|$**.**
**By the property of absolute value, this relation holds true only when** $A$ **and** $B$ **have the same sign, or either of them is zero.**
**That is, the condition is:** $AB \geq 0$
**Substituting values:**
$(x^2 - x)(3x - 15) \geq 0$
$\Rightarrow x(x - 1) \cdot 3(x - 5) \geq 0$
$\Rightarrow 3x(x - 1)(x - 5) \geq 0$
**We use the number line (wavy curve) method to solve the inequality.**
**The critical points are:** $0, 1, 5$
**Checking the signs, we get:**
**Negative (−) for** $x < 0$
**Positive (+) for** $0 \leq x \leq 1$
**Negative (−) for** $1 < x < 5$
**Positive (+) for** $x \geq 5$
**Since the inequality is** $\geq 0$**, the acceptable region is:** $x \in [0, 1] \cup [5, \infty)$
**Comparing with the given set:** $\alpha = 0, \beta = 1, \gamma = 5$
$\therefore \alpha + \beta + \gamma = 0 + 1 + 5 = 6$**.**

**38.** **Let the common root of the two equations be** $\alpha$**.**
**So** $\alpha$ **satisfies both equations:**
$\alpha^2 + c\alpha + d = 0$ **.....(i)**
$\alpha^2 + d\alpha + c = 0$ **.....(ii)**
**Subtracting (ii) from (i):**
$(c - d)\alpha + (d - c) = 0 \Rightarrow (c - d)\alpha - (c - d) = 0 \Rightarrow (c - d)(\alpha - 1) = 0$
**Since** $c \neq d$**, we have** $c - d \neq 0$
$\therefore \alpha - 1 = 0 \Rightarrow \alpha = 1$
**Putting** $\alpha = 1$ **in equation (i):** $1 + c + d = 0 \Rightarrow c + d = -1$
**Product of roots of the 1st equation** $x^2 + cx + d = 0$ **is** $d$
**If one root is** $\alpha = 1$**, let the other (non-common) root be** $\beta_1$**.**
$\therefore 1 \cdot \beta_1 = d \Rightarrow \beta_1 = d$
**Product of roots of the 2nd equation** $x^2 + dx + c = 0$ **is** $c$
**If one root is** $\alpha = 1$**, let the other (non-common) root be** $\beta_2$**.**
$\therefore 1 \cdot \beta_2 = c \Rightarrow \beta_2 = c$
**So the roots of the new equation are** $c$ **and** $d$**.**
**Sum of roots** $= c + d = -1$
**Product of roots** $= cd$
**Required equation:** $x^2 - (c + d)x + cd = 0 \Rightarrow x^2 + x + cd = 0$**.**

**39.** **Given equation:** $x^3 - 5x^2 + 8x - 4 = 0$
**Since the roots of the equation are** $\alpha, \beta, \gamma$**, the sum of the roots:**
$\sum \alpha = \alpha + \beta + \gamma = -(-5) = 5$
**We need to find:** $(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha)$
**We know,**
$\alpha + \beta = 5 - \gamma$
$\beta + \gamma = 5 - \alpha$
$\gamma + \alpha = 5 - \beta$
$\therefore (\alpha + \beta)(\beta + \gamma)(\gamma + \alpha) = (5 - \gamma)(5 - \alpha)(5 - \beta)$
**Let** $f(x) = (x - \alpha)(x - \beta)(x - \gamma)$
**Since the roots of the equation are** $\alpha, \beta, \gamma$ **and the coefficient of** $x^3$ **is** $1$**,**
$(x - \alpha)(x - \beta)(x - \gamma) = x^3 - 5x^2 + 8x - 4$
**We essentially need to find** $f(5)$ **by putting** $x = 5$**.**
$f(5) = (5 - \alpha)(5 - \beta)(5 - \gamma) = 5^3 - 5(5^2) + 8(5) - 4$
$= 125 - 125 + 40 - 4 = 36$
$\therefore (\alpha + \beta)(\beta + \gamma)(\gamma + \alpha) = 36$**.**

**40.** **Since the three roots of the cubic equation are in arithmetic progression (A.P),**
**let the three roots be:** $a - d$**,** $a$**,** $a + d$
**Sum of the roots:**
$(a - d) + a + (a + d) = -\frac{-9}{1}$
$\Rightarrow 3a = 9$
$\Rightarrow a = 3$
**Since** $a = 3$ **is a root of the equation, it satisfies it.**
**Putting** $3$ **in place of** $x$**, we can directly find the value of** $k$**:**
$3^3 - 9(3^2) + k(3) - 24 = 0$
$\Rightarrow 27 - 9(9) + 3k - 24 = 0$
$\Rightarrow 27 - 81 + 3k - 24 = 0$
$\Rightarrow -54 - 24 + 3k = 0$
$\Rightarrow -78 + 3k = 0$
$\Rightarrow 3k = 78$
$\Rightarrow k = 26$
$\therefore$ **The required value of** $k$ **is** $26$**.**
**(Check: putting** $k = 26$**, the roots come out to be** $2, 3, 4$**, which are in arithmetic progression)**

**Chemistry** **200 marks**

**41.** **Let the** $20.0\,\text{mL}$ **portion contain** $NaOH = a\,\text{mmol}$ **and** $Na_2CO_3 = b\,\text{mmol}$**.**
**Up to the phenolphthalein endpoint, all the** $NaOH$ **is neutralized and the** $Na_2CO_3$ **is protonated once to** $NaHCO_3$**, so** $a + b = 12.0 \times 0.150 = 1.80\,\text{mmol}$
**The additional volume to the methyl orange endpoint neutralizes the** $NaHCO_3$ **produced, whose amount equals the original** $Na_2CO_3$**, so** $b = 8.00 \times 0.150 = 1.20\,\text{mmol}$
$a = 1.80 - 1.20 = 0.60\,\text{mmol}$
**In** $200\,\text{mL}$ **(10× scale-up),** $Na_2CO_3 = 12.0\,\text{mmol}$**,** $NaOH = 6.00\,\text{mmol}$
$m(Na_2CO_3) = 12.0 \times 10^{-3} \times 106 = 1.272\,\text{g}$**,** $m(NaOH) = 6.00 \times 10^{-3} \times 40 = 0.240\,\text{g}$
**Total impurity mass** $= 2.00 - 1.272 - 0.240 = 0.488\,\text{g}$
$\%Na_2CO_3 = \frac{1.272}{2.00} \times 100 = 63.6\%$**,** $\%NaOH = \frac{0.240}{2.00} \times 100 = 12.0\%$**,** $\%\text{impurity} = \frac{0.488}{2.00} \times 100 = 24.4\%$

**42.** $n(HCl)_{given} = 50.0 \times 10^{-3} \times 0.800 = 0.0400\,mol$
$n(HCl)_{excess} = n(NaOH) = 25.0 \times 10^{-3} \times 0.400 = 0.0100\,mol$
$n(HCl)_{used} = 0.0400 - 0.0100 = 0.0300\,mol$
$MCO_3 + 2HCl \rightarrow MCl_2 + CO_2 + H_2O$
$n(MCO_3) = \frac{0.0300}{2} = 0.0150\,mol$
$M(MCO_3) = \frac{1.26}{0.0150} = 84\,gmol^{-1}$
$M + 60 = 84 \Rightarrow M = 24\,gmol^{-1}$**; the metal is** $Mg$**.**
$V\,(CO_2) = 0.0150 \times 22.4 = 0.336\,L$

**43.** $Cr_2O_7^{2-} + 6Fe^{2+} + 14H^+ \rightarrow 2Cr^{3+} + 6Fe^{3+} + 7H_2O$
$n(Cr_2O_7^{2-}) = 40.0 \times 10^{-3} \times 0.0250 = 1.00 \times 10^{-3}\,mol$
**Fully oxidizing this amount of** $Cr_2O_7^{2-}$ **requires** $6 \times 1.00 \times 10^{-3} = 6.00 \times 10^{-3}\,mol$ **of** $Fe^{2+}$**.**
$n(Fe^{2+})_{given} = 60.0 \times 10^{-3} \times 0.150 = 9.00 \times 10^{-3}\,mol$
**Since the** $Fe^{2+}$ **given exceeds what is needed,** $K_2Cr_2O_7$ **is the limiting reactant.**
$n(Fe^{2+})_{unreacted} = 9.00 \times 10^{-3} - 6.00 \times 10^{-3} = 3.00 \times 10^{-3}\,mol$
$MnO_4^- + 5Fe^{2+} + 8H^+ \rightarrow Mn^{2+} + 5Fe^{3+} + 4H_2O$
$n(KMnO_4) = \frac{3.00 \times 10^{-3}}{5} = 6.00 \times 10^{-4}\,mol$
$V\,(KMnO_4) = \frac{6.00 \times 10^{-4}}{0.0250} \times 1000 = 24.0\,mL$

**44.** $MnO_2 + 4HCl \rightarrow MnCl_2 + Cl_2 + 2H_2O$
$Cl_2 + 2KI \rightarrow 2KCl + I_2$
$I_2 + 2Na_2S_2O_3 \rightarrow 2NaI + Na_2S_4O_6$
**Overall,** $MnO_2 \equiv Cl_2 \equiv I_2 \equiv 2S_2O_3^{2-}$
$n(S_2O_3^{2-}) = 96.0 \times 10^{-3} \times 0.250 = 0.0240\,mol$
$n(MnO_2) = \frac{0.0240}{2} = 0.0120\,mol$
$m(MnO_2) = 0.0120 \times 87 = 1.044\,g$
$\%MnO_2 = \frac{1.044}{2.61} \times 100 = 40.0\%$
$m(Na_2S_2O_3 \cdot 5H_2O) = 0.0240 \times 248 = 5.952\,g$

**45.** $n(CuSO_4) = \frac{1.60}{160} = 0.0100\,mol$
$m(H_2O) = 2.50 - 1.60 = 0.90\,g$**,** $n(H_2O) = \frac{0.90}{18} = 0.0500\,mol$
$x = \frac{n(H_2O)}{n(CuSO_4)} = \frac{0.0500}{0.0100} = 5$
$\%\text{crystal water} = \frac{0.90}{2.50} \times 100 = 36.0\%$
**The molar mass of** $CuSO_4 \cdot 5H_2O$ **is** $160 + 5(18) = 250\,gmol^{-1}$
**In** $12.50\,g$ **of the hydrated salt,** $n = \frac{12.50}{250} = 0.0500\,mol$**, so** $m(CuSO_4) = 0.0500 \times 160 = 8.00\,g$

**46.** **Let** $NaCl = x\,mol$ **and** $NaBr = y\,mol$ **in the original mixture.**
$58.5x + 103y = 0.300$ **(mass of original mixture)**
$143.5x + 188y = 0.600$ **(mass of precipitate, since** $n(AgCl) = n(NaCl) = x$ **and** $n(AgBr) = n(NaBr) = y$**)**
**Solving the two equations together,** $x = 1.428 \times 10^{-3}\,mol$**,** $y = 2.102 \times 10^{-3}\,mol$
$m(NaCl) = 1.428 \times 10^{-3} \times 58.5 = 0.0835\,g$
$m(NaBr) = 2.102 \times 10^{-3} \times 103 = 0.2165\,g$
$\%NaCl = \frac{0.0835}{0.300} \times 100 = 27.8\%$**,** $\%NaBr = \frac{0.2165}{0.300} \times 100 = 72.2\%$

**47.** **The** $30.0\,\text{mL}$ **absorbed by** $\text{KOH}$ **is the volume of** $CO_2$ **produced,** $V(CO_2) = 30.0\,\text{mL}$
**For a hydrocarbon** $C_xH_y$, $C_xH_y + \left(x + \frac{y}{4}\right) O_2 \rightarrow xCO_2 + \frac{y}{2}H_2O$; **from the volume ratio,** $15x = 30 \Rightarrow x = 2$
**The remaining gas (oxygen)** $= 75.0 - 30.0 = 45.0\,\text{mL}$
$O_2$ **used** $= 90.0 - 45.0 = 45.0\,\text{mL}$
$15\left(2 + \frac{y}{4}\right) = 45 \Rightarrow y = 4$
**The hydrocarbon is** $C_2H_4$**, and the volume of** $O_2$ **used is** $45.0\,\text{mL}$**.**

**48.** $n(H_3PO_4) = 50.0 \times 10^{-3} \times 0.200 = 10.0\,\text{mmol}$, $n(\text{NaOH}) = 75.0 \times 10^{-3} \times 0.200 = 15.0\,\text{mmol}$
**First,** $10.0\,\text{mmol}$ **of** $\text{NaOH}$ **converts all the** $H_3PO_4$ **to** $H_2PO_4^-$**; the remaining** $\text{NaOH} = 15.0 - 10.0 = 5.0\,\text{mmol}$
**The remaining** $\text{NaOH}$ **converts an equal amount of** $H_2PO_4^-$ **to** $HPO_4^{2-}$**, so** $n(H_2PO_4^-) = 10.0 - 5.0 = 5.0\,\text{mmol}$ **and** $n(HPO_4^{2-}) = 5.0\,\text{mmol}$
**Since** $H_2PO_4^-$ **and** $HPO_4^{2-}$ **are present in equal amounts, this is a buffer, and** $\text{pH} = \text{p}K_{a2} + \log\frac{[HPO_4^{2-}]}{[H_2PO_4^-]} = \text{p}K_{a2} = 7.20$
**Reaching the second equivalence point (all** $HPO_4^{2-}$**) requires a total of** $2 \times n(H_3PO_4) = 20.0\,\text{mmol}$ **of** $\text{NaOH}$
**Additional** $\text{NaOH}$ **needed** $= 20.0 - 15.0 = 5.0\,\text{mmol}$
$V(\text{NaOH})_{\text{additional}} = \frac{5.0}{0.200} = 25.0\,\text{mL}$

**49.** $n(KMnO_4)_{\text{given}} = 50.0 \times 10^{-3} \times 0.200 = 0.0100\,\text{mol}$
$MnO_4^- + 5Fe^{2+} + 8H^+ \rightarrow Mn^{2+} + 5Fe^{3+} + 4H_2O$
**From the** $FeSO_4$ **used to neutralize the excess** $KMnO_4$, $n(KMnO_4)_{\text{excess}} = \frac{100.0\times10^{-3}\times0.200}{5} = 0.00400\,\text{mol}$
$n(KMnO_4)_{\text{reacted}} = 0.0100 - 0.00400 = 0.00600\,\text{mol}$**, which reacted with the** $FeC_2O_4$**.**
$5FeC_2O_4 + 3MnO_4^- + 24H^+ \rightarrow 5Fe^{3+} + 10CO_2 + 3Mn^{2+} + 12H_2O$
$n(FeC_2O_4 \cdot 2H_2O) = \frac{5}{3} \times 0.00600 = 0.0100\,\text{mol}$
**The molar mass of** $FeC_2O_4 \cdot 2H_2O \approx 180\,\text{gmol}^{-1}$**, so** $m = 0.0100 \times 180 = 1.80\,\text{g}$
**Purity** $= \frac{1.80}{2.00} \times 100 = 90.0\%$
$n(CO_2) = 2 \times n(FeC_2O_4 \cdot 2H_2O) = 0.0200\,\text{mol}$, $V(CO_2) = 0.0200 \times 22400 = 448\,\text{mL}$
**The** $Fe^{3+}$ **produced from oxidation of the original** $FeC_2O_4$ **equals** $n(FeC_2O_4 \cdot 2H_2O) = 0.0100\,\text{mol}$

**50.** $MnO_4^- + 5Fe^{2+} + 8H^+ \rightarrow Mn^{2+} + 5Fe^{3+} + 4H_2O$
**In the first aliquot, only** $Fe^{2+}$ **reacts:** $n(Fe^{2+}) = 5 \times (0.0200 \times 0.0100) = 1.00 \times 10^{-3}\,\text{mol}$
**In the second aliquot, all the iron is present as** $Fe^{2+}$**, so the total iron is** $n(Fe)_{\text{total}} = 5 \times (0.0200 \times 0.0250) = 2.50 \times 10^{-3}\,\text{mol}$
$n(Fe^{3+}) = n(Fe)_{\text{total}} - n(Fe^{2+}) = 2.50 \times 10^{-3} - 1.00 \times 10^{-3} = 1.50 \times 10^{-3}\,\text{mol}$
$Fe^{2+} : Fe^{3+} = 1.00 : 1.50 = 2 : 3$
**The scale-up factor from** $25.0\,\text{mL}$ **to one litre is** $\frac{1000}{25.0} = 40$
$n(FeSO_4)_{\text{per litre}} = 1.00 \times 10^{-3} \times 40 = 0.0400\,\text{mol}$, $[FeSO_4] = 0.0400 \times 152 = 6.08\,\text{gL}^{-1}$
$n\{Fe_2(SO_4)_3\}_{\text{per litre}} = \frac{1.50\times10^{-3}}{2} \times 40 = 0.0300\,\text{mol}$, $[Fe_2(SO_4)_3] = 0.0300 \times 400 = 12.0\,\text{gL}^{-1}$

**51.** $MnO_4^- + 5Fe^{2+} + 8H^+ \rightarrow Mn^{2+} + 5Fe^{3+} + 4H_2O$
$n(Fe^{2+}) = 150 \times 10^{-3} \times 0.100 = 0.0150\,mol$
$n(KMnO_4) = \frac{0.0150}{5} = 3.00 \times 10^{-3}\,mol$
$m(KMnO_4)_{pure} = 3.00 \times 10^{-3} \times 158.0 = 0.4740\,g$
**Purity** $= \frac{0.4740}{0.500} \times 100 = 94.8\%$
$m_{impurity} = 0.500 - 0.4740 = 0.0260\,g$
$n(Mn^{2+}) = n(KMnO_4) = 3.00 \times 10^{-3}\,mol$

**52.** **Taking** $1\,L$ **of the concentrated solution, its mass** $= 1000 \times 1.80 = 1800\,g$
$m(H_2SO_4) = 0.980 \times 1800 = 1764\,g$
$M_{conc} = \frac{1764}{98} = 18.0\,M$
**From the dilution formula,** $M_{dilute} = \frac{18.0 \times 12.0}{600} = 0.360\,M$
**Since** $H_2SO_4$ **is diprotic,** $n(NaOH) = 2 \times n(H_2SO_4) = 2 \times 0.360 \times 25.0 \times 10^{-3} = 0.0180\,mol$
$V(NaOH) = \frac{0.0180}{0.450} \times 1000 = 40.0\,mL$

**53.** **EDTA forms a** $1{:}1$ **complex with both** $Ca^{2+}$ **and** $Mg^{2+}$**.**
$n(EDTA) = 12.5 \times 10^{-3} \times 0.0100 = 1.25 \times 10^{-4}\,mol$
$n(CaCO_3)_{equiv} = 1.25 \times 10^{-4}\,mol$**,** $m(CaCO_3) = 1.25 \times 10^{-4} \times 100 = 1.25 \times 10^{-2}\,g = 12.5\,mg$
**Concentration in the** $50.0\,mL$ **sample** $= \frac{12.5\,mg}{50.0 \times 10^{-3}\,L} = 250\,mgL^{-1} = 250\,ppm$
**French degree,** $= \frac{250}{10} = 25.0°f$
**In** $2.00\,L$ **of sample,** $CaCO_3$**-equivalent mass** $= 250 \times 2.00 = 500\,mg$

**54.** $m(C) = 0.264 \times \frac{12}{44} = 0.0720\,g$**,** $m(H) = 0.108 \times \frac{2}{18} = 0.0120\,g$
$m(O) = 0.120 - 0.0720 - 0.0120 = 0.0360\,g$
$\%C = \frac{0.0720}{0.120} \times 100 = 60\%$**,** $\%H = \frac{0.0120}{0.120} \times 100 = 10\%$**,** $\%O = \frac{0.0360}{0.120} \times 100 = 30\%$
**The mole ratio of elements is** $C : H : O = \frac{60}{12} : \frac{10}{1} : \frac{30}{16} = 5 : 10 : 1.875$
**Dividing by the smallest value and multiplying by** $3$ **gives the whole-number ratio** $C : H : O = 8 : 16 : 3$
**The empirical formula is** $C_8H_{16}O_3$**, with mass** $= 8(12) + 16(1) + 3(16) = 160\,gmol^{-1}$**, matching the given molar mass.**
**The empirical and molecular formulas coincide:** $C_8H_{16}O_3$**.**

**55.** **Converting** $H_3PO_4$ **to** $NaH_2PO_4$ **removes 1 proton, so** $E_1 = \frac{98}{1} = 98\,gmol^{-1}$
**Converting to** $Na_2HPO_4$ **removes 2 protons,** $E_2 = \frac{98}{2} = 49\,gmol^{-1}$
**Converting to** $Na_3PO_4$ **removes 3 protons,** $E_3 = \frac{98}{3} = 32.7\,gmol^{-1}$
$n(H_3PO_4) = \frac{4.90}{98} = 0.0500\,mol$
**For neutralization to** $Na_2HPO_4$**,** $n(NaOH) = 2 \times 0.0500 = 0.100\,mol$**,** $V_2 = \frac{0.100}{0.500} \times 1000 = 200\,mL$
**For neutralization to** $Na_3PO_4$**,** $n(NaOH) = 3 \times 0.0500 = 0.150\,mol$**,** $V_3 = \frac{0.150}{0.500} \times 1000 = 300\,mL$

**56.** $n(O_2) = \frac{8.00}{32} = 0.250\,mol$**,** $n(N_2) = \frac{14.0}{28} = 0.500\,mol$
$n_{total} = 0.750\,mol$
$X_{O_2} = \frac{0.250}{0.750} = \frac{1}{3}$**,** $X_{N_2} = \frac{0.500}{0.750} = \frac{2}{3}$
**From Dalton's law of partial pressures,** $P_{O_2} = X_{O_2} \times P_{total} = \frac{1}{3} \times 2.00 = 0.667\,atm$
$P_{N_2} = \frac{2}{3} \times 2.00 = 1.333\,atm$
$\overline{M} = \frac{m_{total}}{n_{total}} = \frac{8.00 + 14.0}{0.750} = 29.33\,gmol^{-1}$
$\varrho_{STP} = \frac{\overline{M}}{22.4} = \frac{29.33}{22.4} = 1.31\,gL^{-1}$

**57.** $CaOCl_2 + H_2SO_4 \rightarrow CaSO_4 + H_2O + Cl_2$
$Cl_2 + 2KI \rightarrow 2KCl + I_2$
$I_2 + 2Na_2S_2O_3 \rightarrow 2NaI + Na_2S_4O_6$
$n(I_2) = \frac{0.0320 \times 0.250}{2} = 4.00 \times 10^{-3}$ mol
$n(Cl_2) = n(I_2) = 4.00 \times 10^{-3}$ mol
$m(Cl_2) = 4.00 \times 10^{-3} \times 71 = 0.284$ g
**Available chlorine** $= \frac{0.284}{1.20} \times 100 = 23.67\%$
$V\,(Cl_2)_{STP} = 4.00 \times 10^{-3} \times 22400 = 89.6$ mL

**58.** $KHC_8H_4O_4$ **is a monoprotic acid, so** $n(NaOH) = n(KHP)$
$n(KHP) = \frac{0.5105}{204.2} = 2.500 \times 10^{-3}$ mol
$M(NaOH) = \frac{2.500 \times 10^{-3}}{25.00 \times 10^{-3}} = 0.1000$ M
$n(NaOH)_{20mL} = 0.1000 \times 20.00 \times 10^{-3} = 2.000 \times 10^{-3}$ mol
**From** $NaOH + HCl \rightarrow NaCl + H_2O$**,** $n(HCl) = n(NaOH)_{20mL} = 2.000 \times 10^{-3}$ mol
$M(HCl) = \frac{2.000 \times 10^{-3}}{18.50 \times 10^{-3}} = 0.1081$ M

**59.** **On heating, only** $NaHCO_3$ **decomposes:** $2NaHCO_3 \rightarrow Na_2CO_3 + H_2O + CO_2$**; the mass loss per mole of** $NaHCO_3$ **is** $\frac{62}{2} = 31$ g**.**
**Let** $m(NaHCO_3) = x$ g **and** $m(Na_2CO_3) = y$ g**;** $x + y = 16.44$
$\frac{x}{84} \times 31 = 3.72 \Rightarrow x = 10.08$ g
$y = 16.44 - 10.08 = 6.36$ g
**The total** $Na_2CO_3$ **in the residue after heating is** $n = \frac{6.36}{106} + \frac{1}{2}\left(\frac{10.08}{84}\right) = 0.0600 + 0.0600 = 0.1200$ mol
**In the** 20.0 mL **portion,** $n(Na_2CO_3) = 0.1200 \times \frac{20.0}{200} = 0.0120$ mol
**Up to phenolphthalein, the** $Na_2CO_3$ **is protonated once, so** $n(HCl) = n(Na_2CO_3) = 0.0120$ mol
$V_{phenolphthalein} = \frac{0.0120}{0.150} \times 1000 = 80.0$ mL
**The additional volume to the methyl orange endpoint neutralizes the same amount, so the additional volume is also** 80.0 mL**.**

**60.** **Let** $n(CaCO_3) = x$ mol **and** $n(MgCO_3) = y$ mol**;** $100x + 84y = 10.20$
$CaCO_3 \rightarrow CaO + CO_2$ **and** $MgCO_3 \rightarrow MgO + CO_2$**; the mass loss equals the** $CO_2$ **produced,** $44(x + y) = 4.84 \Rightarrow x + y = 0.110$
**Solving the two equations,** $x = 0.0600$ mol**,** $y = 0.0500$ mol
$m(CaCO_3) = 0.0600 \times 100 = 6.00$ g**,** $m(MgCO_3) = 0.0500 \times 84 = 4.20$ g
$\%CaCO_3 = \frac{6.00}{10.20} \times 100 = 58.82\%$**,** $\%MgCO_3 = \frac{4.20}{10.20} \times 100 = 41.18\%$
$m(CaO) = 0.0600 \times 56 = 3.36$ g**,** $m(MgO) = 0.0500 \times 40 = 2.00$ g
**Total residue** $= 3.36 + 2.00 = 5.36$ g
**Dissolving each oxide (**$CaO + 2HCl \rightarrow CaCl_2 + H_2O$**, similarly for** $MgO$**) requires** 2 mol HCl **per mole of oxide,** $n(HCl) = 2(0.0600 + 0.0500) = 0.220$ mol
$V\,(HCl) = \frac{0.220}{1.00} \times 1000 = 220$ mL