MCQ 50 marks

1. **D.** $(-\infty,-3)\cup(-\sqrt{2},-1)\cup(1,\sqrt{2})\cup(3,\infty)$

Explanation: Let $t = x^2$. The base condition is $t > 1$, $t \neq 2$; the argument is positive when $t < 4$ or $t > 9$

For $1 < t < 2$ the base lies between 0 and 1 and $0 < \frac{t-4}{t-9} < 1$; so the logarithm is positive

For $t > 2$ the square root needs $\frac{t-4}{t-9} \geq 1$; but $\frac{t-4}{t-9} - 1 = \frac{5}{t-9}$, so $t > 9$

Hence $1 < x^2 < 2$ or $x^2 > 9$; that is, $1 < |x| < \sqrt{2}$ or $|x| > 3$

All endpoints of both cases are excluded (the base becomes 0 or 1, or the denominator vanishes)

Therefore, the domain is $(-\infty,-3)\cup(-\sqrt{2},-1)\cup(1,\sqrt{2})\cup(3,\infty)$

2. **B.** $(-\infty,-2]\cup(2,\infty)$

Explanation: Let $t = \frac{x^2-1}{x^2+1} = 1 - \frac{2}{x^2+1}$; then $t \in [-1,0)\cup(0,1)$ and $f(x) = t + \frac{1}{t}$

For $-1 \leq t < 0$, the range of $t + \frac{1}{t}$ is $(-\infty,-2]$; the value -2 is attained at $t = -1$, i.e. $x = 0$

For $0 < t < 1$ the range is $(2,\infty)$; getting the value 2 would need $t = 1$, which is impossible

Each branch is continuous and tends to the stated limits

Therefore, the full range is $(-\infty,-2]\cup(2,\infty)$

3. **B.** $[2\sqrt{5},\infty)$

Explanation: The given expression is $\sqrt{(x-2)^2+1} + \sqrt{(-x-2)^2+1}$

By the vector triangle inequality it is $\geq \sqrt{((x-2)+(-x-2))^2+(1+1)^2} = 2\sqrt{5}$

Equality needs $(x-2, 1)$ and $(-x-2, 1)$ to be proportional in the same direction; so $x = 0$, and indeed $f(0) = 2\sqrt{5}$

f is continuous and tends to infinity as $x \to \infty$

Therefore, the range is $[2\sqrt{5},\infty)$

4. **D.** $-1 < k \leq 0$

Explanation: Let $t = \log_2 x \in [-2,3]$ and $u = |t|$; the equation becomes $(u-2)^2 = k+1$

For $0 < u \leq 2$ each u gives two values of t; for $2 < u \leq 3$ it gives one t; for $u = 0$ it gives one t

For $-1 < k \leq 0$, $u = 2 - \sqrt{k+1} \in [1,2)$ and $u = 2 + \sqrt{k+1} \in (2,3]$; so there are $2 + 1 = 3$ solutions

At $k = -1$ there are two; for $0 < k < 3$ two; at $k = 3$ one; otherwise none

Therefore, the required condition is $-1 < k \leq 0$

5. **A.** $\sqrt{\frac{1+\sqrt{2}}{2}}$

Explanation: Putting $x = y = \frac{t}{2}$, $f(t) + 1 = 2f\left(\frac{t}{2}\right)^2$; so $f(t) \geq -1$ for all real t

From $t = 1$, $f\left(\frac{1}{2}\right)^2 = \frac{3+1}{2} = 2$; the negative branch $-\sqrt{2} < -1$ is impossible, so $f\left(\frac{1}{2}\right) = \sqrt{2}$

From $t = \frac{1}{2}$, $f\left(\frac{1}{4}\right)^2 = \frac{1+\sqrt{2}}{2}$

The negative square root would be below -1, which is impossible

Therefore, $f\left(\frac{1}{4}\right) = \sqrt{\frac{1+\sqrt{2}}{2}}$

6. **B.** $\frac{685}{6}$

Explanation: The first values are $f(0) = 2$, $f(1) = -\frac{1}{3}$, $f(2) = -\frac{3}{2}$, $f(3) = 2$

The same recurrence makes these three values repeat cyclically; no denominator vanishes

The sum of every three terms is $2 - \frac{1}{3} - \frac{3}{2} = \frac{1}{6}$

The number of terms is $2027 = 3 \times 675 + 2$; so after 675 full cycles the terms $2, -\frac{1}{3}$ remain

Therefore, the sum is $\frac{675}{6} + 2 - \frac{1}{3} = \frac{685}{6}$

7. **A.** 4

Explanation: P is non-zero; let $P(x) = x^m Q(x)$ with $Q(0) \neq 0$. Then $Q(x^2) = Q(x)^2$, so $Q(0) = Q(0)^2 \Rightarrow Q(0) = 1$

If Q is non-constant, let its term of lowest positive degree be cx^r with $c \neq 0$

The coefficient of x^r in $Q(x)^2$ is $2c$; but in $Q(x^2)$ that coefficient is zero, since Q has no positive power below r; this is impossible

So $Q = 1$, i.e. $P(x) = x^m$

Now $2^m = 16 \Rightarrow m = 4$

8. **D.** $\frac{25}{4}$

Explanation: From the second relation, the odd part of f is $x^3 + 2x$; so $f(x) = x^3 + ax^2 + 2x + b$

Substituting in the first relation, $(2a+3)x^2 - (2a+3)x + (3+a+2b) = 2$
Comparing coefficients, $a = -\frac{3}{2}$ and $3 - \frac{3}{2} + 2b = 2 \Rightarrow b = \frac{1}{4}$
Therefore, $f(2) = 8 + 4a + 4 + b = 8 - 6 + 4 + \frac{1}{4} = \frac{25}{4}$

9. **A.** 0
Explanation: If $f(a) = f(b)$, applying f again gives $-a = -b \Rightarrow a = b$; so f is one-one
A continuous one-one function on the real line is strictly increasing or strictly decreasing
In both cases $f \circ f$ is strictly increasing
But $x \mapsto -x$ is strictly decreasing
Therefore, no such continuous f exists; the number is 0

10. **B.** $-\frac{1}{8}$
Explanation: Let $f(x) = ax^2 + bx + c$; in $f(f(x))$ the coefficients of x^4 and x^3 are a^3 and $2a^2b$ respectively
$a^3 = 8 \Rightarrow a = 2$ and $2a^2b = -24 \Rightarrow b = -3$
The coefficient of x^2 is $a(b^2 + 2ac) + ab = 12 + 8c = 20$, so $c = 1$; with these values the remaining coefficients also match
So $f(x) = 2x^2 - 3x + 1 = 2\left(x - \frac{3}{4}\right)^2 - \frac{1}{8}$
Therefore, the minimum value is $-\frac{1}{8}$

11. **D.** $\frac{1+\sqrt{5}}{2} < a < 2$
Explanation: For an upward parabola to have two distinct roots in $(0, 3)$: the discriminant is positive, the vertex abscissa lies in $(0, 3)$, and the values at both ends are positive
The discriminant is $4(a^2 - a - 1) > 0$; combined with $a > 0$ this gives $a > \frac{1+\sqrt{5}}{2}$
At $x = 0$ the value is $a + 1 > 0$, automatic; at $x = 3$ the value is $10 - 5a > 0 \Rightarrow a < 2$
Under these conditions the vertex condition $0 < a < 3$ holds
Therefore, $\frac{1+\sqrt{5}}{2} < a < 2$

12. **B.** $\frac{9}{4}$
Explanation: $f_a(x) = (x - a)^2 - a^2$
For $-1 \le a \le 2$ the minimum is $-a^2$ and the maximum occurs at the endpoint farther from the vertex; so $L(a) = \max\{(a+1)^2, (a-2)^2\}$
The larger of these two distances is smallest at the midpoint $a = \frac{1}{2}$; then $L = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$
For $a < -1$ the function is increasing, so $L = (4 - 4a) - (1 + 2a) = 3 - 6a > 9$
For $a > 2$ it is decreasing, so $L = (1 + 2a) - (4 - 4a) = 6a - 3 > 9$
Therefore, the overall minimum is $\frac{9}{4}$

13. **C.** 4
Explanation: For $x \neq \pm 1$, $f(x) = \pm(x^2 - 1)$ and $f(x) \neq 0$
By continuity the choice of sign is constant on each of $(-\infty, -1)$, $(-1, 1)$, $(1, \infty)$; otherwise f would have to vanish in between
Since $f(0) = 1$, on the middle interval $f(x) = 1 - x^2$
On each outer interval there are two independent choices
At $x = \pm 1$ both formulas give zero, so every choice joins continuously; the total is $2^2 = 4$ functions

14. **B.** $0 < k < 2$
Explanation: Let $h(x) = x^3 - 3x$; $h'(x) = 3(x^2 - 1)$, so h is increasing, decreasing and increasing on $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$ respectively
The local maximum is $h(-1) = 2$ and the local minimum is $h(1) = -2$; so for $-2 < c < 2$, $h(x) = c$ has three distinct roots
For $0 < k < 2$, each of $h(x) = k$ and $h(x) = -k$ has three roots, and the two sets of roots are disjoint
At $k = 0$ there are three roots, at $k = 2$ four, for $k > 2$ two, and for $k < 0$ none
Therefore, the required condition is $0 < k < 2$

15. **A.** $\mathbb{R} \setminus \left\{-1, 0, \frac{1}{2}, 1\right\}$
Explanation: $f(x) = \frac{x-1}{x-3}$, but in the original domain $x \neq 2, 3$. For the composite we also need $f(x) \neq 2, 3$
$f(x) = 2$ gives $x = 5$ and $f(x) = 3$ gives $x = 4$; so the domain of the composite is $\mathbb{R} \setminus \{2, 3, 4, 5\}$
Simplifying, $f(f(x)) = \frac{1}{4-x}$; excluding only $x \neq 4$ its range would be $\mathbb{R} \setminus \{0\}$
The extra excluded points $x = 2, 3, 5$ remove the values $\frac{1}{2}, 1, -1$ respectively; since $\frac{1}{4-x}$ is one-one, no other input brings them back
Therefore, the range is $\mathbb{R} \setminus \left\{-1, 0, \frac{1}{2}, 1\right\}$

16. **B.** $e^{\frac{1}{e}}$

Explanation: At a common point, $y = a^x$ and $x = a^y$; since the exponential function is increasing, neither $x > y$ nor $x < y$ is possible; so $x = y$

Hence $a^x = x \Rightarrow \ln a = \frac{\ln x}{x}$, where $x > 1$

For $H(x) = \frac{\ln x}{x}$, $H'(x) = \frac{1-\ln x}{x^2}$; it attains its unique maximum $\frac{1}{e}$ at $x = e$

To meet the horizontal line $y = \ln a$ in exactly one point we need $\ln a = \frac{1}{e}$

Therefore, $a = e^{\frac{1}{e}}$

17. B. $\frac{40}{27}$

Explanation: Let $u = f^{-1}(t)$ and $v = f^{-1}(8 - t)$; then $u + v = 2$ and $u^3 + v^3 + u + v = 8$

$u^3 + v^3 = (u + v)^3 - 3uv(u + v) = 8 - 6uv$; so $8 - 6uv + 2 = 8 \Rightarrow uv = \frac{1}{3}$

u, v are the two distinct real roots of $z^2 - 2z + \frac{1}{3} = 0$

f is strictly increasing; so $t = f(u)$ and $t = f(v)$ are two distinct solutions

The required product is $f(u)f(v) = uv(u^2 + 1)(v^2 + 1) = uv\left((uv)^2 + (u + v)^2 - 2uv + 1\right)$

$= \frac{1}{3}\left(\frac{1}{9} + 4 - \frac{2}{3} + 1\right) = \frac{1}{3} \times \frac{40}{9} = \frac{40}{27}$

18. C. $-\frac{45}{4}$

Explanation: $f(x)f(-x) = \sqrt{\left(\sqrt{x^2+1} + x\right)\left(\sqrt{x^2+1} - x\right)} = 1$ and both values are positive

Let $u = f(x)$; then $f(-x) = \frac{1}{u}$ and $u + \frac{1}{u} = 3 \Rightarrow u - \frac{1}{u} = \pm\sqrt{9-4} = \pm\sqrt{5}$

Also $f(x)^2 - f(-x)^2 = 2x$

So $2x = \left(u - \frac{1}{u}\right)\left(u + \frac{1}{u}\right) = \pm 3\sqrt{5}$

$x = \pm\frac{3\sqrt{5}}{2}$; both values are acceptable

Therefore, the product is $-\frac{45}{4}$

19. C. $\frac{1}{3}$

Explanation: $f(f(x)) = \sqrt{1 - (1 - x^2)} = |x|$

So successive composition alternates $f, |x|, f, |x|, \ldots$; since 2026 is even, $f^{\circ 2026}(x) = |x|$

The equation becomes $|x| = \frac{x}{2} + \frac{1}{4}$

For $x \geq 0$, $x = \frac{1}{2}$; for $x < 0$, $-x = \frac{x}{2} + \frac{1}{4} \Rightarrow x = -\frac{1}{6}$

Both values lie in the domain

Therefore, the sum is $\frac{1}{2} - \frac{1}{6} = \frac{1}{3}$

20. A. $e^{\frac{9}{2}}$

Explanation: Let $t = \ln x$ and $s = \ln\frac{e^3}{x} = 3 - t$; the equation is $\frac{t}{1+t^2} = \frac{s}{1+s^2}$

Both denominators are positive; cross-multiplying gives $(t - s)(1 - ts) = 0$

In the first case $t = \frac{3}{2}$; in the second $t(3 - t) = 1 \Rightarrow t = \frac{3\pm\sqrt{5}}{2}$; the three values are distinct

So the product of the roots x is $e^{\frac{3}{2}+\frac{3+\sqrt{5}}{2}+\frac{3-\sqrt{5}}{2}} = e^{\frac{9}{2}}$

21. C. $\frac{5}{4} < a < 3$

Explanation: For $x \leq 0$ the left side is positive while the right side is non-positive; so $x > 0$

Dividing the left side by x gives $\frac{7}{x} - 3$, $\frac{5}{x} - 1$, $1 + \frac{1}{x}$, $3 - \frac{7}{x}$ on $(0, 1], (1, 2], (2, 4], (4, \infty)$ respectively

The first three parts together decrease strictly on $(0, 4]$ from infinity down to $\frac{5}{4}$; the last part increases from $\frac{5}{4}$ but never reaches 3

So for $\frac{5}{4} < a < 3$ there are two distinct roots, one on each branch

At $a = \frac{5}{4}$ there is one root, and for $a \geq 3$ only the first branch gives a single root

22. D. -1

Explanation: Near any real a we can take sequences of both rational and irrational numbers

For continuity the limits of the two branches must agree: $a^2 = 2 - a \Rightarrow (a + 2)(a - 1) = 0$

At $a = -2, 1$ both branches tend to the same value and $f(a)$ equals it; so f is truly continuous there

Therefore, the required sum is $-2 + 1 = -1$

23. D. -2

Explanation: $f(x + 6) = -f(x + 3) = f(x)$; so 6 is a period

$f(1) = f(-2 + 3) = -f(-2)$ and since f is even, $f(2) = f(-2) = -f(1) = -2$

Let $f(0) = c$; then $f(3) = -c$, $f(4) = -f(1) = -2$, $f(5) = -f(2) = 2$

One six-term cycle is $c, 2, -2, -c, -2, 2$; its sum is zero

The number of terms is $2027 = 6 \times 337 + 5$; after the full cycles the terms $c, 2, -2, -c, -2$ remain

Therefore, the sum is -2, which does not depend on the unknown c

24. **A.** 3

Explanation: If $P(x) = c$ is constant, the condition is $c = c^2$, so $c = 0$ or $c = 1$

If P is non-constant, its range contains infinitely many real values; for each such $t = P(x)$, $P(t) - t^2 = 0$

A non-zero polynomial cannot have infinitely many roots; so $P(t) \equiv t^2$

$P = 0$, $P = 1$ and $P(x) = x^2$ — all three satisfy the condition

Therefore, the number is 3

25. **D.** 486

Explanation: The sum of all digits is 12. If the sum of the digits in the odd positions is S, then $2S - 12$ must be divisible by 11

$2S - 12$ is even and $-12 \le 2S - 12 \le 8$, so $2S - 12 = 0 \Rightarrow S = 6$

The digits in the odd positions can only be $\{1, 1, 2, 2\}$, $\{0, 1, 2, 3\}$ or $\{0, 0, 3, 3\}$

First case: arrangements of the odd and even positions give $6 \times 6 = 36$

Second case: with a non-zero first digit, $(4! - 3!) \times 4! = 18 \times 24 = 432$; third case: $(6 - 3) \times 6 = 18$

Therefore, the total is $36 + 432 + 18 = 486$

26. **C.** 552

Explanation: For divisibility by 15 the last digit must be 0 or 5 and the digit sum must be divisible by 3; the sum of all seven digits is 21

So the omitted digit must be 0, 3 or 6

Omitting 0: the last digit is 5 and the other five digits can be arranged in $5! = 120$ ways

Omitting 3 or 6: last digit 0 gives $5! = 120$ numbers; last digit 5 gives 4 non-zero choices for the first place and then 4! ways, i.e. 96 numbers

Therefore, the total is $120 + 2(120 + 96) = 552$

27. **B.** 28800

Explanation: Treat A, B as one block; its internal orders number 2. Excluding C, D, E, there are 6 objects including this block

These six objects can be seated around the circle in 5! ways; there are 6 gaps between them along the circle

C, D, E must go into three different gaps; the number of ways to choose gaps and place the persons is ${}^6C_3 \times 3!$

Therefore, the total is $2 \times 5! \times {}^6C_3 \times 3! = 28800$

28. **C.** 312

Explanation: First seat the five men in 4! ways (rotation removed). The wives go into the five gaps between them; for each wife the two gaps beside her husband are forbidden

These 10 forbidden wife–gap relations form a 10-cycle; the number of ways to choose k forbidden relations is $r_k = \frac{10}{10-k}\,{}^{10-k}C_k$

For $k = 0, 1, \ldots, 5$, $r_k = 1, 10, 35, 50, 25, 2$

By inclusion–exclusion, the valid arrangements of the wives number $5! - 10 \cdot 4! + 35 \cdot 3! - 50 \cdot 2! + 25 \cdot 1! - 2 \cdot 0! = 13$

Therefore, the total is $4! \times 13 = 312$

29. **D.** $m = \pm 4$

Explanation: At a common point, $2x^2 = mx - 2 \Rightarrow 2x^2 - mx + 2 = 0$

For exactly one common point the discriminant of this quadratic must be zero

$m^2 - 4(2)(2) = 0 \Rightarrow m^2 = 16 \Rightarrow m = \pm 4$

In both cases there is exactly one point of tangency

30. **B.** 24

Explanation: First arrange C, B, N in $3! = 6$ ways

These three letters leave 4 gaps, including the two ends

To keep the A's apart, the three identical A's go into three different gaps; the number of ways is ${}^4C_3 = 4$

Therefore, the total is $6 \times 4 = 24$

31. **D.** 5040

Explanation: There are 4 S's, 2 E's and one each of A, M, N, T

Treating the three vowels A, E, E as one block gives $4 + 1 + 1 + 1 + 1 = 8$ objects (four S's, M, N, T and the block)

The block has $\frac{3!}{2!} = 3$ internal arrangements

Therefore, the total is $\frac{8!}{4!} \times 3 = 5040$

32. **A.** 23520

Explanation: The consonants S, S, S, T, T, T, C can be arranged in $\frac{7!}{3!\,3!} = 140$ ways
This creates 8 gaps, including the two ends; choosing distinct gaps for the three vowels can be done in ${}^{8}C_{3} = 56$ ways
In the chosen gaps, A, I, I can be placed in $\frac{3!}{2!} = 3$ ways
Therefore, the total is $140 \times 56 \times 3 = 23520$

33. **C.** 1320
Explanation: The units digit must be 0 or 5
If the units digit is 0, the number of ways to arrange five of the remaining six digits is ${}^{6}P_{5} = 720$
If the units digit is 5, there are 5 choices for the first digit (excluding zero); the middle four places can be filled from the remaining five digits in ${}^{5}P_{4} = 120$ ways
Therefore, the total is $720 + 5 \times 120 = 1320$

34. **A.** 12600
Explanation: The consonants N, N, N, G, G, R can be arranged in $\frac{6!}{3!\,2!} = 60$ ways
This creates 7 gaps, including the two ends; choosing distinct gaps for the five vowels can be done in ${}^{7}C_{5} = 21$ ways
In the chosen gaps, E, E, E, I, I can be placed in $\frac{5!}{3!\,2!} = 10$ ways
Therefore, the total is $60 \times 21 \times 10 = 12600$

35. **A.** 1260
Explanation: The two mentioned pairs are two 2-cycles. The remaining four elements can have no fixed point and no further 2-cycle
So the remaining four elements must form a single 4-cycle
The number of ways to choose two unnamed pairs is $\frac{{}^{8}C_{2}\,{}^{6}C_{2}}{2!} = \frac{28 \times 15}{2} = 210$; the number of 4-cycles on the remaining four elements is $(4-1)! = 6$
Therefore, the total number of functions is $210 \times 6 = 1260$

36. **D.** 6720
Explanation: First, keeping only A, B together: choose their team in 3 ways, their partners in ${}^{10}C_{2}$ ways, and distribute the rest between the other two teams in ${}^{8}C_{4}$ ways; total 9450
Now remove the invalid cases in which C, D are also together
If all four are in the same team, the count is $3 \times {}^{8}C_{4} = 210$
If the two pairs are in different teams, choose the teams in 3×2 ways and the other members in ${}^{8}C_{2} \times {}^{6}C_{2}$ ways; the count is 2520
Therefore, the valid divisions number $9450 - 210 - 2520 = 6720$

37. **D.** 40
Explanation: If the first student gets 1, 3, 5, the other three together get 13, 11, 9 respectively
For each of the other three, let $y_i = 5 - x_i$; then $0 \le y_i \le 4$ and the sums are 2, 4, 6 respectively
For sums 2 and 4 the numbers of ways are ${}^{4}C_{2} = 6$ and ${}^{6}C_{2} = 15$
For sum 6, ignoring the upper bound gives ${}^{8}C_{2} = 28$; each case with some $y_i \ge 5$ removes ${}^{3}C_{2} = 3$, so the valid count is $28 - 3 \times 3 = 19$
Therefore, the total is $6 + 15 + 19 = 40$

38. **A.** 61
Explanation: Let $x = u + 1$, $y = u + v + 2$, $z = u + v + w + 3$, where $u, v, w \ge 0$ are integers
Then $3u + 2v + w = 24$ and $0 \le u \le 8$
For a given u, the number of choices of v is $\left\lfloor \frac{24-3u}{2} \right\rfloor + 1$; for each choice w is determined
Therefore, the total is $13 + 11 + 10 + 8 + 7 + 5 + 4 + 2 + 1 = 61$

39. **C.** 153
Explanation: From the chosen numbers $a_1 < \cdots < a_5$ let $b_i = a_i - (i-1)$; then $1 \le b_1 < \cdots < b_5 \le 11$, and this transformation is one-to-one $\sum a_i = \sum b_i + 10$; for divisibility by 3, the remainder of $\sum b_i$ on division by 3 must be 2
In 1 to 11, the numbers with remainder 0, 1, 2 are 3, 4, 4 in count
The remainder-wise count triples (n_0, n_1, n_2) can be
(0, 2, 3), (1, 0, 4), (1, 3, 1), (2, 1, 2), (3, 2, 0)
The numbers of ways are ${}^{3}C_{n_0}\,{}^{4}C_{n_1}\,{}^{4}C_{n_2}$, namely 24, 3, 48, 72, 6
Therefore, the total is $24 + 3 + 48 + 72 + 6 = 153$

40. **A.** 28

Explanation: The two interior returns split the path into three segments; each segment touches the diagonal only at its two ends
If a segment has m right and m up steps, removing its first right step and last up step gives a path staying below the diagonal; so the number of segments is $C_{m-1} = \frac{1}{m}\,{}^{2m-2}C_{m-1}$
The three segment sizes are positive with sum 6; the possible size patterns are $(1,1,4),(1,2,3),(2,2,2)$
Here $C_0 = 1$, $C_1 = 1$, $C_2 = 2$, $C_3 = 5$
So the corresponding counts are $3 \cdot 1 \cdot 1 \cdot 5 = 15$, $6 \cdot 1 \cdot 1 \cdot 2 = 12$ and 1
Therefore, the total number of paths is $15 + 12 + 1 = 28$

41. **C.** 8

Explanation: Six 1's in three blocks with at least two each means every block has exactly two 1's
So the six 0's go into the end gaps (between 0 and 2 each) and the two inner gaps (between 1 and 2 each), since three consecutive 0's are not allowed
Inner gaps $(1,1)$: the ends must be $(2,2)$: 1 way
Inner gaps $(1,2)$ or $(2,1)$: the ends are $(1,2)$ or $(2,1)$: 2 ways each, 4 in all
Inner gaps $(2,2)$: the ends are $(0,2),(1,1),(2,0)$: 3 ways
Therefore, the total number of sequences is $1 + 4 + 3 = 8$

42. **C.** 8

Explanation: Under a symmetry group, the number of distinct arrangements is the average of the numbers of arrangements fixed by each symmetry; here there are 8 rotations and 8 reflections, 16 symmetries in all
Rotations contribute: the zero rotation fixes ${}^8C_4 = 70$; rotations by 2 and 6 places fix 2 each; the rotation by 4 places fixes 6; odd rotations fix 0; the total is $70 + 2 + 2 + 6 = 80$
The 4 reflections whose axis passes through two opposite beads fix 6 each (two fixed positions and three pairs)
The other 4 reflections, with the axis through gaps, have four pairs; ${}^4C_2 = 6$ fixed arrangements each
Therefore, the number of necklaces is $\frac{80+4\times6+4\times6}{16} = 8$

43. **A.** 40

Explanation: Let a, b, c be the numbers of sides between consecutive chosen vertices around the circle; then $a, b, c \geq 1$ and $a + b + c = 12$
The corresponding angles of the triangle are $\frac{a\pi}{12}, \frac{b\pi}{12}, \frac{c\pi}{12}$; for an acute triangle we need $a, b, c \leq 5$
Putting $u = 5 - a$, $v = 5 - b$, $w = 5 - c$ gives $u + v + w = 3$; there are ${}^5C_2 = 10$ non-negative solutions, and $a, b, c \geq 1$ holds automatically
The first vertex can be chosen in 12 ways, but each triangle is counted from each of its three vertices
Therefore, the total number of triangles is $\frac{12\times10}{3} = 40$

44. **C.** 1176

Explanation: Choose the two children who receive gifts in ${}^4C_2 = 6$ ways; between them, A, B can be given separately in 2 ways
The child to whom C, D go together can be chosen in 2 ways; so after the first four gifts the two children hold 3 and 1
The remaining six distinct gifts can be distributed in $2^6 = 64$ ways; both children already have at least one gift
To get an equal 5 each, the child holding 3 must receive 2 more; ${}^6C_2 = 15$ such distributions are removed
Therefore, the total number of valid distributions is $6 \times 2 \times 2 \times (64 - 15) = 1176$

45. **D.** 3360

Explanation: Since $f^{\circ 3}$ is the identity, f is a permutation; every cycle length divides 3, i.e. is 1 or 3
Choose the three fixed elements in ${}^9C_3 = 84$ ways; the remaining six form two 3-cycles
The number of ways to form two unnamed groups of three is $\frac{6!}{(3!)^2\,2!} = 10$; each group has $(3 - 1)! = 2$ three-cycles
Therefore, the total number of functions is $84 \times 10 \times 2^2 = 3360$

46. **B.** 156

Explanation: Each row has $3! = 6$ choices; without restriction the total is $6^3 = 216$
If a particular column has a single letter, the letter can be chosen in 3 ways and the other two cells of each row can be filled in 2 ways; so the count is $3 \cdot 2^3 = 24$
If two particular columns are each single-lettered, their letters must differ; the choices number $3 \cdot 2 = 6$, and then the third column is also single-lettered
Therefore, by inclusion–exclusion the valid count is $216 - 3 \cdot 24 + 3 \cdot 6 - 6 = 156$

47. **C.** 22
Explanation: The number has the form abccba, where $a \in \{1, \dots, 9\}$ and $b, c \in \{0, \dots, 9\}$
For divisibility by 4 we need $10b + a$ divisible by 4; so $a = 2, 4, 6, 8$; for $a = 2, 6$, b is odd and for $a = 4, 8$, b is even; there are 20 pairs (a, b) in all
For divisibility by 9 the digit sum $2(a + b + c)$ must be divisible by 9, i.e. $a + b + c$ must be divisible by 9
Each pair gives one c; when $a + b$ leaves remainder 0 on division by 9, both $c = 0$ and $c = 9$ work
Among the allowed pairs, $a + b = 9$ only for $(2, 7)$ and $(6, 3)$
Therefore, the total is $20 + 2 = 22$

48. **A.** 122
Explanation: If C, D form the two-member group, the other six can be divided into sizes $1, 2, 3$ in $\frac{6!}{1!\,2!\,3!} = 60$ ways
In this case A, B together is invalid: 4 ways if in the two-member group, $4 \times 3 = 12$ if in the three-member group; the valid count is $60 - 4 - 12 = 44$
If C, D are in the three-member group, the third member can be A or B (2 choices); the other five can be divided as $1, 2, 2$ in $\frac{5!}{1!\,(2!)^2\,2!} = 15$ ways; that is 30 valid divisions
If the third member is one of the other four, then in each case 3 of the 15 have A, B paired and are removed; the valid count is $4(15 - 3) = 48$
Therefore, the total number of divisions is $44 + 30 + 48 = 122$

49. **B.** -252
Explanation: The coefficient of x^{10} in $(1 - x)^{10}(1 + x)^{10}$ is $\sum_{r=0}^{10}(-1)^r\, {}^{10}C_r\, {}^{10}C_{10-r}$
Since ${}^{10}C_{10-r} = {}^{10}C_r$, this coefficient is exactly the required sum
But the product equals $(1 - x^2)^{10}$; the coefficient of $x^{10} = (x^2)^5$ in it is $(-1)^5\, {}^{10}C_5$
Therefore, the required value is -252

50. **B.** 1920
Explanation: The common value taken by $1, 2$ can be chosen in 4 ways; its pre-image has odd size at least 2; since the other three values are also used, this size can only be 3 or 5
Size 5: choose three more elements from the remaining six in ${}^6C_3$ ways; the other three map one-to-one to the other three values; the count is $4 \cdot {}^6C_3 \cdot 3! = 480$
Size 3: choose the extra element in 6 ways; the remaining five split into three parts of sizes $3, 1, 1$
Choose which value receives the part of size 3 in 3 ways, pick its three elements in ${}^5C_3 = 10$ ways, and distribute the other two in 2! ways; the count is $4 \cdot 6 \cdot 3 \cdot 10 \cdot 2 = 1440$
Therefore, the total number of functions is $480 + 1440 = 1920$