Engineering Daily 21 [Set 03] EV - Answer Sheet

Functions and Graphs of Functions + Permutations and Combinations

Total Marks: 50 **Negative Marks:** 0.25 **Time:** 1 Hour

MCQ 50 marks

1. **B.** $(-\infty, -2) \cup [1, 4)$

Explanation: For the square root to be real, $\frac{x-1}{x+2} \geq 0$ and $x \neq -2$

By sign analysis, $x \in (-\infty, -2) \cup [1, \infty)$

For the logarithm, $4 - x > 0$

$\Rightarrow x < 4$

Therefore, the domain is $(-\infty, -2) \cup [1, 4)$

2. **C.** $(1, 2) \cup (2, 5)$

Explanation: For the base of the logarithm, $x - 1 > 0$ and $x - 1 \neq 1$

That is, $x > 1$ and $x \neq 2$

Also, $5 - x > 0$

$\Rightarrow x < 5$

Therefore, the domain is $(1, 2) \cup (2, 5)$

3. **A.** $(1, 3]$

Explanation: $f(x) = \frac{(x-2)^2+3}{(x-2)^2+1} = 1 + \frac{2}{(x-2)^2+1}$

Let $t = (x - 2)^2 + 1$, so $t \geq 1$

$\therefore 0 < \frac{2}{t} \leq 2$

Since t takes every value in $[1, \infty)$, $\frac{2}{t}$ takes every value in $(0, 2]$

Therefore, the range is $(1, 3]$

4. **B.** $[2, 2\sqrt{2}]$

Explanation: The domain is $[1, 5]$

Let $u = x - 1$ and $v = 5 - x$; then $u, v \geq 0$ and $u + v = 4$

$f(x)^2 = u + v + 2\sqrt{uv} = 4 + 2\sqrt{uv}$

Also, $0 \leq uv \leq \left(\frac{u+v}{2}\right)^2 = 4$

$\Rightarrow 4 \leq f(x)^2 \leq 8$

The value is 2 at the endpoints and $2\sqrt{2}$ at $x = 3$; the function is continuous

Therefore, the range is $[2, 2\sqrt{2}]$

5. **D.** $\frac{x+3}{2-x}$

Explanation: Let $y = \frac{2x-3}{x+1}$

$\Rightarrow xy + y = 2x - 3$

$\Rightarrow x(y - 2) = -(y + 3)$

$\Rightarrow x = \frac{y+3}{2-y}$

Therefore, $f^{-1}(x) = \frac{x+3}{2-x}$

6. **D.** $(-\infty, 2) \cup [\frac{7}{3}, \infty)$

Explanation: $(f \circ g)(x) = \sqrt{3 - \frac{1}{x-2}} = \sqrt{\frac{3x-7}{x-2}}$

Here, $x \neq 2$ and $\frac{3x-7}{x-2} \geq 0$

The sign-change points are 2 and $\frac{7}{3}$

Therefore, the domain is $(-\infty, 2) \cup [\frac{7}{3}, \infty)$

7. **A.** $x^2 - 4x + 2$

Explanation: Given, $f(x) + 2f(1 - x) = 3x^2$ ... (i)

Replacing x by $1 - x$, $f(1 - x) + 2f(x) = 3(1 - x)^2$ ...(ii)

From $2\times$(ii)$-$(i), $3f(x) = 6(1 - x)^2 - 3x^2$

$\Rightarrow f(x) = 2(1 - x)^2 - x^2$

Therefore, $f(x) = x^2 - 4x + 2$

8. **D.** $\frac{5}{2}$

Explanation: Let $t = \frac{x-1}{x+1}$

$1 + t^2 = \frac{(x+1)^2+(x-1)^2}{(x+1)^2} = \frac{2(x^2+1)}{(x+1)^2}$

$\therefore f(t) = \frac{1+t^2}{2}$

$t = 2$ is allowed (it occurs at $x = -3$)

Therefore, $f(2) = \frac{1+4}{2} = \frac{5}{2}$

9. **C.** 1

Explanation: $f(f(x)) = \frac{a(ax+3)+3(2x-1)}{2(ax+3)-(2x-1)} = \frac{(a^2+6)x+3a-3}{(2a-2)x+7}$

From $f(f(x)) = x$, $(2a - 2)x^2 + (1 - a^2)x - 3a + 3 = 0$

This holds for infinitely many x, so $2a - 2 = 0$

$\Rightarrow a = 1$

For $a = 1$, $f(f(x)) = \frac{7x}{7} = x$ and the function is non-constant

(For $a = -6$, $f(x) = -3$, which is constant)

Therefore, $a = 1$

10. **D.** $2 + \sqrt{x - 3}$

Explanation: Let $y = x^2 - 4x + 7 = (x - 2)^2 + 3$

$\Rightarrow (x - 2)^2 = y - 3$

In the domain $x \geq 2$, so $x - 2 \geq 0$

$\Rightarrow x = 2 + \sqrt{y - 3}$

Therefore, $f^{-1}(x) = 2 + \sqrt{x - 3}$, where $x \geq 3$

11. **D.** $(1, 6)$

Explanation: To get the input 2 in the new graph, $3 - x = 2$

$\Rightarrow x = 1$
Then the new $y = -2f(2) + 4 = -2(-1) + 4 = 6$
Therefore, the transformed point is $(1, 6)$

12. **A.** $0 < k < 1$
Explanation: From $|(x-2)^2 - 1| = k$, $(x-2)^2 = 1 + k$ or $(x-2)^2 = 1 - k$
For four distinct roots, both right-hand sides must be positive and different from each other
Therefore, $k > 0$ and $1 - k > 0 \Rightarrow 0 < k < 1$
(At $k = 0$ there are two roots and at $k = 1$ there are three distinct roots)

13. **B.** 2
Explanation: For $a \geq 0$, $|x| = 2 + a$ or $|x| = 2 - a$
For $0 < a < 2$ there are four roots; for $a = 0$ there are two roots; for $a > 2$ there are two roots
For $a = 2$, $|x| = 4$ or $|x| = 0$, so the roots are $-4, 0, 4$
Therefore, $a = 2$

14. **A.** 18
Explanation: At an intersection point, $x^2 = 2|x| + 3$
Let $t = |x| \geq 0$
$t^2 - 2t - 3 = 0$
$\Rightarrow (t-3)(t+1) = 0$
Since $t \geq 0$, $t = 3$, so $x = \pm 3$
Therefore, the required sum is $(-3)^2 + 3^2 = 18$

15. **B.** $(-\infty, 2]$
Explanation: $f(x) = (x-a)^2 + 3 - a^2$; the x-coordinate of the vertex is a
If $a \leq 2$, then $x - a \geq 0$ on $[2, \infty)$, so f is strictly increasing and one-one
If $a > 2$, taking $0 < \varepsilon < a - 2$ gives $a - \varepsilon$, $a + \varepsilon \in [2, \infty)$ but $f(a - \varepsilon) = f(a + \varepsilon)$
Therefore, the necessary and sufficient condition is $a \leq 2$

16. **A.** $(-1, 1)$
Explanation: For $x \geq 0$, $f(x) = \frac{x}{1+x} = 1 - \frac{1}{1+x}$; the range of this branch is $[0, 1)$
For $x < 0$, $f(x) = \frac{x}{1-x} = -1 + \frac{1}{1-x}$; the range of this branch is $(-1, 0)$
Therefore, the full range is $(-1, 0) \cup [0, 1) = (-1, 1)$

17. **B.** $\frac{1}{2}\log_2\left(\frac{1+x}{1-x}\right)$
Explanation: Let $y = \frac{2^x - 2^{-x}}{2^x + 2^{-x}} = \frac{2^{2x}-1}{2^{2x}+1}$
$\Rightarrow y(2^{2x} + 1) = 2^{2x} - 1$
$\Rightarrow (1 - y)2^{2x} = 1 + y$
$\Rightarrow 2^{2x} = \frac{1+y}{1-y}$
$\Rightarrow x = \frac{1}{2}\log_2\left(\frac{1+y}{1-y}\right)$, which is defined since $-1 < y < 1$
Therefore, $f^{-1}(x) = \frac{1}{2}\log_2\left(\frac{1+x}{1-x}\right)$

18. **C.** $(-\infty, -2] \cup [2, \infty)$
Explanation: The domain is $x > 0$, $x \neq 1$
Let $t = \log_2 x$; then $t \in \mathbb{R} \setminus \{0\}$ and $f(x) = t + \frac{1}{t}$
If $t > 0$, $t + \frac{1}{t} \geq 2$; if $t < 0$, $t + \frac{1}{t} \leq -2$
Conversely, $t + \frac{1}{t} = y \Rightarrow t^2 - yt + 1 = 0$ has a real nonzero root when $y^2 - 4 \geq 0$
Therefore, the range is $(-\infty, -2] \cup [2, \infty)$

19. **D.** $\frac{1}{2}$
Explanation: $(f \circ g)(x) = (2x+1)^2 - 1 = 4x^2 + 4x$
$(g \circ f)(x) = 2(x^2 - 1) + 1 = 2x^2 - 1$
Given, $4x^2 + 4x = 2x^2 - 1$
$\Rightarrow 2x^2 + 4x + 1 = 0$
Its discriminant is $16 - 8 = 8 > 0$, so the roots are real
Therefore, the product of the roots is $\frac{1}{2}$

20. **A.** $[-1, \infty)$
Explanation: Adding the two equations, $2f(x) = 2x^2 + 4x$
$\Rightarrow f(x) = x^2 + 2x = (x+1)^2 - 1$
$(x+1)^2$ takes all non-negative values and equals 0 at $x = -1$
Therefore, the range is $[-1, \infty)$

21. **D.** 0
Explanation: Let $t = x^2 \geq 0$; then $(t-2)^2 = k + 4$
For $k < -4$ there is no root; for $k = -4$, $t = 2$ gives two roots
For $-4 < k < 0$, two positive values of t give four roots; for $k > 0$, one positive t gives two roots
For $k = 0$, $x^2(x^2 - 4) = 0$, i.e. $x = -2, 0, 2$
Therefore, $k = 0$ gives exactly three intersection points

22. **C.** 4

Explanation: The domain is $[-1, 3]$
$|f(x) - 1| = \frac{1}{2} \Rightarrow f(x) = \frac{1}{2}$ or $f(x) = \frac{3}{2}$
In the first case $(x-1)^2 = 4 - \frac{1}{4} = \frac{15}{4}$, so $x = 1 \pm \frac{\sqrt{15}}{2}$
In the second case $(x-1)^2 = 4 - \frac{9}{4} = \frac{7}{4}$, so $x = 1 \pm \frac{\sqrt{7}}{2}$
All four values are distinct and lie in the domain
Therefore, there are 4 intersection points

23. **C.** $[3, \infty)$
Explanation: By the triangle inequality, $|x^2 - 1| + |x^2 - 4| \geq |(x^2 - 1) - (x^2 - 4)| = 3$
For $1 \leq x^2 \leq 4$, $f(x) = (x^2 - 1) + (4 - x^2) = 3$
For $x^2 \geq 4$, $f(x) = 2x^2 - 5$, which takes every value from 3 upward
Therefore, the range is $[3, \infty)$

24. **B.** $\mathbb{R} \setminus \{2\}$
Explanation: The domain of the original expression is $\mathbb{R} \setminus \{1\}$
It simplifies to $f(x) = x + 1$, but the condition $x \neq 1$ remains
To get $f(x) = 2$ we would need $x = 1$, which is not in the domain
For any other $y \neq 2$, we can take $x = y - 1 \neq 1$
Therefore, the range is $\mathbb{R} \setminus \{2\}$

25. **A.** $m = \pm 2$
Explanation: At a common point, $x^2 = mx - 1 \Rightarrow x^2 - mx + 1 = 0$
For exactly one common point the discriminant of this quadratic must be zero
$m^2 - 4 = 0 \Rightarrow m = \pm 2$
In both cases there is exactly one point of tangency

26. **B.** 12
Explanation: First arrange B, N, N in $\frac{3!}{2!} = 3$ ways
These three letters leave 4 gaps, including the two ends
To keep the A's apart, the three identical A's go into three different gaps; the number of ways is ${}^4C_3 = 4$
Therefore, the total number of arrangements is $3 \times 4 = 12$

27. **D.** 840
Explanation: Here there are 4 I's, 4 S's, 2 P's and 1 M
Treating the four identical I's as one block, the total number of objects is $1 + 4 + 2 + 1 = 8$
The block has only 1 internal arrangement
Therefore, the total number of arrangements is $\frac{8!}{4!\,2!} = 840$

28. **C.** 1058400
Explanation: The consonants M, M, T, T, H, C, S can be arranged in $\frac{7!}{2!\,2!} = 1260$ ways
This creates 8 gaps, including the two ends
Choosing distinct gaps for the four vowels can be done in ${}^8C_4 = 70$ ways
In the chosen gaps, A, A, E, I can be placed in $\frac{4!}{2!} = 12$ ways
Therefore, the total number of arrangements is $1260 \times 70 \times 12 = 1058400$

29. **C.** 216
Explanation: The units digit must be 0 or 5
If the units digit is 0, the number of ways to arrange four of the remaining five digits is ${}^5P_4 = 120$
If the units digit is 5, there are 4 choices for the first digit (excluding zero); the middle three places can be filled from the remaining four digits in ${}^4P_3 = 24$ ways
Therefore, the total is $120 + 4 \times 24 = 216$

30. **D.** 192
Explanation: For divisibility by 4, the number formed by the last two digits must be divisible by 4
The allowed endings are
12, 16, 24, 32, 36, 52, 56, 64; that is 8 in all
For each ending, the remaining four digits fill the first four places in $4! = 24$ ways
Therefore, the total is $8 \times 24 = 192$

31. **C.** 43200
Explanation: First seat the boys in 5! ways; this creates 6 gaps, including the two ends
Choose 4 gaps for the girls in ${}^6C_4$ ways
Seat the four girls in the chosen gaps in 4! ways
Therefore, the total is $5! \times {}^6C_4 \times 4! = 120 \times 15 \times 24 = 43200$

32. **A.** 72

Explanation: Unrestricted circular arrangements = $(6-1)! = 120$
Treating the two specified persons as one block gives 5 objects, with $(5-1)!$ circular arrangements
The two persons can swap inside the block in 2! ways, so the number of ways they sit together is $4! \times 2 = 48$
Therefore, the number of ways they are not together is $120 - 48 = 72$

33. **B.** 350
Explanation: The possible splits are 2 men and 3 women, or 3 men and 2 women
First case: ${}^{6}C_2 \times {}^{5}C_3 = 15 \times 10 = 150$
Second case: ${}^{6}C_3 \times {}^{5}C_2 = 20 \times 10 = 200$
Therefore, the total number of committees is $150 + 200 = 350$

34. **A.** 480
Explanation: Choose exactly one of A, B in 2 ways and three of the other six in ${}^{6}C_3 = 20$ ways
So the number of committees is $2 \times 20 = 40$
In each committee, two different persons can be made president and secretary in ${}^{4}P_2 = 12$ ways
Therefore, the total is $40 \times 12 = 480$

35. **B.** 146
Explanation: Unrestricted committees = ${}^{10}C_5 = 252$
Committees with A, B together: ${}^{8}C_3 = 56$; with C, D together: also 56
Committees with all four: ${}^{6}C_1 = 6$; these are removed twice
By inclusion–exclusion, the required number is $252 - 56 - 56 + 6 = 146$

36. **B.** 5775
Explanation: If the groups were labelled, the number of divisions would be $\frac{12!}{(4!)^3}$
Since the groups are unnamed, each division has been counted 3! times (once for each order of the groups)
Therefore, the required number is $\frac{12!}{(4!)^3\,3!} = 5775$

37. **A.** 35
Explanation: If the chosen numbers are $a_1 < a_2 < a_3 < a_4$, then $a_{i+1} \geq a_i + 2$
Let $b_1 = a_1$, $b_2 = a_2 - 1$, $b_3 = a_3 - 2$, $b_4 = a_4 - 3$
Then $1 \leq b_1 < b_2 < b_3 < b_4 \leq 7$, and every such choice corresponds to exactly one valid original selection
Therefore, the required number is ${}^{7}C_4 = 35$

38. **B.** 110
Explanation: The set has 6 even and 6 odd numbers
The sum of three numbers is even if all are even, or if two are odd and one is even
First case: ${}^{6}C_3 = 20$; second case: ${}^{6}C_2 \times {}^{6}C_1 = 15 \times 6 = 90$
Therefore, the total number of selections is $20 + 90 = 110$

39. **C.** 7420
Explanation: Choose the two letters that go to the correct envelopes in ${}^{8}C_2 = 28$ ways
None of the remaining six may go to the correct envelope; the number of such arrangements is $D_6 =$
$6!\left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right)$
$D_6 = 360 - 120 + 30 - 6 + 1 = 265$
Therefore, the total is ${}^{8}C_2 \times D_6 = 28 \times 265 = 7420$

40. **A.** 80
Explanation: Taking the numbers in the order of colours as x_1, x_2, x_3, x_4, we need $x_1 + x_2 + x_3 + x_4 = 10$ with $1 \leq x_i \leq 6$
Ignoring the upper bound, the number of positive integer solutions is ${}^{9}C_3 = 84$
If a particular $x_i \geq 7$, subtracting 6 from it gives positive solutions of sum 4; the count is ${}^{3}C_3 = 1$
Two variables cannot both be 7 or more
Therefore, the total number of outcomes is $84 - 4 \times 1 = 80$

41. **B.** 66
Explanation: In 9 steps we must go 4 right and 5 up; the total number of paths is ${}^{9}C_4 = 126$
Paths from $(0, 0)$ to $(2, 2)$: ${}^{4}C_2 = 6$
Paths from $(2, 2)$ to $(4, 5)$: ${}^{5}C_2 = 10$; so the paths through the forbidden point number 6 ×

$10 = 60$
Therefore, the required number of paths is $126 - 60 = 66$

42. **B.** 42
Explanation: Unrestricted paths: ${}^{10}C_5 = 252$
Swapping right and up steps after a bad path first reaches $y = x + 1$ gives a one-to-one match with paths from $(0, 0)$ to $(4, 6)$; the reverse transformation also works
So the bad paths number ${}^{10}C_4 = 210$
Therefore, the required number is $252 - 210 = 42$

43. **D.** 115
Explanation: Unrestricted selections of three points: ${}^{10}C_3 = 120$
Choosing three of the four points on the first line gives no triangle; there are ${}^{4}C_3 = 4$ such selections
Choosing the three points on the second line gives ${}^{3}C_3 = 1$ more invalid selection
The two groups are separate and no other three points are collinear
Therefore, the number of triangles is $120 - 4 - 1 = 115$

44. **A.** 2170
Explanation: Total selections: ${}^{15}C_5 = 3003$
Selections with no red, no blue or no green: ${}^{11}C_5 = 462$, ${}^{10}C_5 = 252$, ${}^{9}C_5 = 126$ respectively
With two colours absent, only green gives ${}^{6}C_5 = 6$ and only blue gives ${}^{5}C_5 = 1$; only red is impossible
By inclusion–exclusion, the number is $3003 - 462 - 252 - 126 + 6 + 1 = 2170$

45. **C.** 5720
Explanation: We know that $r\,{}^{8}C_r = 8\,{}^{7}C_{r-1}$
So the given sum is $8\sum_{r=1}^{5} {}^{7}C_{r-1}\,{}^{6}C_{5-r} = 8\sum_{s=0}^{4} {}^{7}C_s\,{}^{6}C_{4-s}$
Adding all the ways of choosing 4 people in total from two separate groups of 7 and 6 gives ${}^{13}C_4$
Therefore, the required value is $8 \times {}^{13}C_4 = 8 \times 715 = 5720$

46. **C.** 2880
Explanation: Choose the adjacent pair in ${}^{3}C_2 = 3$ ways, with 2 internal orders
Treating the pair as a block gives 6 objects; unrestricted arrangements: 6!
If the third specified book is next to the block, placing it just before or just after the block gives $2 \times 5!$ forbidden arrangements
Therefore, the total is $3 \times 2 \times (6! - 2 \times 5!) = 6(720 - 240) = 2880$

47. **C.** 390
Explanation: Since $f(1), f(2)$ are different, their ordered choice can be made in $3 \times 2 = 6$ ways
Once these two values are fixed, the third element of B must occur among $f(3), f(4), f(5), f(6)$ at least once
The unrestricted choices for the remaining four values number 3^4; those in which the third element never occurs number 2^4
Therefore, the required number is $6(3^4 - 2^4) = 6(81 - 16) = 390$

48. **A.** 10
Explanation: We know that ${}^{n}P_r = r!\,{}^{n}C_r$
So $r! = \frac{360}{15} = 24 \Rightarrow r = 4$
Now ${}^{n}C_4 = 15$; for $n = 6$, ${}^{6}C_4 = 15$
For $n \geq 4$, ${}^{n}C_4$ is strictly increasing, so the only value is $n = 6$
Therefore, $n + r = 6 + 4 = 10$

49. **D.** $(0, \infty)$
Explanation: Let $g(x) = \frac{5x^3 - 2x^2}{x - 4} = \frac{x^2(5x-2)}{x-4}$
For $\ln g(x)$ to be defined, $g(x) > 0$
So, $x \neq 0$ and $\frac{5x-2}{x-4} > 0$, that is $x < \frac{2}{5}$ or $x > 4$
Again, $f(x) = e^{\ln g(x)} = g(x)$
Here, as $x \to 0$ or $x \to \frac{2}{5}$, $g(x) \to 0^+$, and as $x \to -\infty$, $g(x) \to \infty$
Again, $g(x)$ is continuous, so on $(-\infty, 0)$ it takes every positive value
Therefore, the range is $(0, \infty)$

50. **A.** 900000
Explanation: Let the 6-digit number be $d_1d_2d_3d_4d_5d_6$
Here, the first digit d_1 can be chosen in 9 ways (excluding 0)
and, each of the other 5 digits can be chosen in 10 ways
So, total numbers $= 9 \times 10^5$
Therefore, total numbers = 900000