MCQ 50 marks

1. **D.** 5

Explanation: We know, the maximum value of $a\cos x + b\sin x$ is $\sqrt{a^2+b^2}$

Here, $a = 3$ and $b = 4$

So, the maximum value $= \sqrt{3^2+4^2}$

$= \sqrt{25}$

Therefore, the maximum value = 5

2. **C.** $(-\infty, -\sqrt{5}) \cup (\sqrt{5}, \infty)$

Explanation: For the outer logarithm to be defined, $\log_3(x^2-4) > 0$

Since the base $3 > 1$, $x^2 - 4 > 1 \Rightarrow x^2 > 5$

Under this condition the argument of the inner logarithm is also positive

Therefore, the domain is $(-\infty, -\sqrt{5}) \cup (\sqrt{5}, \infty)$

3. **A.** $(1, \infty)$

Explanation: The conditions from the logarithm and the square root together give $0 < \frac{x-1}{x+1} \le 1$, since $0 < \frac{1}{2} < 1$

The ratio is positive when $x < -1$ or $x > 1$

$\frac{x-1}{x+1} = 1 - \frac{2}{x+1}$; for $x < -1$ its value exceeds 1, so that part is excluded

For $x > 1$ the ratio lies between 0 and 1

Therefore, the domain is $(1, \infty)$

4. **C.** $\left[\frac{1}{2+\sqrt{2}}, \frac{1}{2-\sqrt{2}}\right]$

Explanation: $\sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$, so $-\sqrt{2} \le \sin x + \cos x \le \sqrt{2}$

$2 - \sqrt{2} \le 2 + \sin x + \cos x \le 2 + \sqrt{2}$ and the whole interval is positive

Taking reciprocals, $\frac{1}{2+\sqrt{2}} \le f(x) \le \frac{1}{2-\sqrt{2}}$

Both end values and every value between them are attained, so this is the range

5. **C.** $\frac{1+x^2}{1-x^2}$

Explanation: Let $y = \sqrt{\frac{x-1}{x+1}}$, where $0 \le y < 1$

$y^2(x+1) = x - 1$

$\Rightarrow x(1-y^2) = 1 + y^2$

$\Rightarrow x = \frac{1+y^2}{1-y^2}$; this value is always at least 1 and unique

Therefore, $f^{-1}(x) = \frac{1+x^2}{1-x^2}$, $0 \le x < 1$

6. **B.** e^2

Explanation: Both composite functions are defined when $x > 0$

The equation is $x^2 = e^{(\ln x)^2}$

Taking natural logarithms on both sides,

$2\ln x = (\ln x)^2$

Let $t = \ln x$; then $t(t-2) = 0 \Rightarrow t = 0$ or $t = 2$

So the solutions are $x = 1, e^2$

Therefore, the required product is e^2

7. **C.** $\frac{2}{3}$

Explanation: Putting $x = 2, -1, \frac{1}{2}$ in turn:

$f(2) + 2f(-1) = 2$ ...(i)

$f(-1) + 2f\left(\frac{1}{2}\right) = -1$ ...(ii)

$f\left(\frac{1}{2}\right) + 2f(2) = \frac{1}{2}$ ...(iii)

From (ii), $f(-1) = -1 - 2f\left(\frac{1}{2}\right)$; substituting into (i), $f(2) - 4f\left(\frac{1}{2}\right) = 4$

From (iii), $f\left(\frac{1}{2}\right) = \frac{1}{2} - 2f(2)$

$\therefore f(2) - 4\left(\frac{1}{2} - 2f(2)\right) = 4 \Rightarrow 9f(2) = 6$

Therefore, $f(2) = \frac{2}{3}$

8. **D.** 7

Explanation: The first branch maps one-to-one onto $(-\infty, 0]$ and the second one-to-one onto $(0, \infty)$; so the inverse exists

Since $-8 \le 0$, $f^{-1}(-8) = \sqrt[3]{-8} = -2$

Since $3 > 0$, $f^{-1}(3) = 3^2 = 9$

Therefore, the required value is $-2 + 9 = 7$

9. **C.** $(-\infty, -1] \cup [1, \infty)$

Explanation: On the branch $x \ge 0$, $x = ax + 1 \Rightarrow x = \frac{1}{1-a}$; this branch has a root exactly when $a < 1$

On the branch $x < 0$, $-x = ax + 1 \Rightarrow x = -\frac{1}{1+a}$; this branch has a root exactly when $a > -1$

If $-1 < a < 1$, each branch has one root, giving two roots in all

If $a \le -1$ or $a \ge 1$, only one branch has a root

Therefore, the required set is $(-\infty, -1] \cup [1, \infty)$

10. **C.** $\left[0, \frac{3}{2}\right]$

Explanation: Let $t = 2x - 1$

On the first branch, $2t + 4 \ge 2$ gives $-1 \le t \le 0$

On the second branch, $4 - t \ge 2$ gives $0 < t \le$

2

So $f(t) \geq 2$ exactly when $-1 \leq t \leq 2$

$-1 \leq 2x - 1 \leq 2 \Rightarrow 0 \leq x \leq \frac{3}{2}$

11. **D.** 7

Explanation: From the first relation, $f(4 - t) = f(t)$, and from the second, $f(10 - t) = f(t)$

Hence $f(t + 6) = f(10 - (4 - t)) = f(4 - t) = f(t)$; that is, 6 is a period

Since $2025 = 6 \times 337 + 3$, $f(2025) = f(3)$

Using the first symmetry, $f(3) = f(1) = 7$

12. **D.** $\left(-\infty, -\frac{3}{2}\right] \cup \left[-\frac{1}{2}, \infty\right)$

Explanation: The input condition is $-3 \leq \frac{x-1}{x+1} \leq 5$, where $x \neq -1$

Substituting $\frac{x-1}{x+1} = 1 - \frac{2}{x+1}$ gives $-4 \leq -\frac{2}{x+1} \leq 4$

That is, $\left|\frac{2}{x+1}\right| \leq 4 \Rightarrow |x + 1| \geq \frac{1}{2}$

Therefore, $x \leq -\frac{3}{2}$ or $x \geq -\frac{1}{2}$; $x = -1$ is already absent from this set

13. **B.** $(-1, 1)$

Explanation: For an odd function, $f(0) = 0$; so $\frac{1+a}{1+b} = 0 \Rightarrow a = -1$

Let $t = e^x > 0$; then $f(x) + f(-x) = \frac{t-1}{t+b} + \frac{1-t}{1+bt}$

On simplifying, the sum is $\frac{(b-1)(t-1)^2}{(t+b)(1+bt)}$

For this to vanish for all $t > 0$, $b = 1$

Therefore, $(a, b) = (-1, 1)$

14. **A.** 3

Explanation: f is strictly increasing and onto, so h exists; since $f(-x) = -f(x)$, $h(-y) = -h(y)$

Hence $h(7) + h(-7) = 0$

$f(3) = 3^3 + 3 = 30 \Rightarrow h(30) = 3$

Therefore, the required value is $0 + 3 = 3$

15. **D.** 2

Explanation: g is a composition of three one-one onto functions, so g^{-1} exists

$g(x) = 5 \Rightarrow f^{-1}(2f(x) + 1) = 5 \Rightarrow 2f(x) + 1 = f(5) = 7$

$\Rightarrow f(x) = 3 = f(2)$

Since f is one-one, $x = 2$

Therefore, $g^{-1}(5) = 2$

16. **D.** $x = 1$

Explanation: Let $t = f(x) = x + \frac{1}{x} \geq 2$

$f(t) = \frac{5}{2} \Rightarrow t + \frac{1}{t} = \frac{5}{2} \Rightarrow (2t - 1)(t - 2) = 0$

Since $t \geq 2$, only $t = 2$ is acceptable

$x + \frac{1}{x} = 2 \Rightarrow (x - 1)^2 = 0$

Therefore, $x = 1$

17. **B.** $\{0\} \cup [3, \infty)$

Explanation: For $k < 0$ there is no solution; for $k = 0$, $2^x = 3$ gives one solution

For $k > 0$, $2^x = 3 + k$ or $2^x = 3 - k$; the first equation always has one solution

The second has a solution exactly when $k < 3$; at $k = 3$, $2^x = 0$ is impossible

Therefore, exactly one solution occurs when $k = 0$ or $k \geq 3$

18. **D.** $\mathbb{R} \setminus \{-1, 0, 1\}$

Explanation: For the inner $f(x)$ to be defined, $x \neq \pm 1$

For the outer f we also need $f(x) \neq \pm 1$

$f(x) = 1$ gives $x^2 + 1 = x^2 - 1$, which is impossible

$f(x) = -1$ gives $x^2 + 1 = -(x^2 - 1) \Rightarrow x = 0$; this value must be excluded

Therefore, the domain is $\mathbb{R} \setminus \{-1, 0, 1\}$

19. **B.** $(1, 3)$

Explanation: In the domain $x \neq 0, 1, 3$; each fraction equals 1 if the expression inside the modulus is positive and -1 if it is negative

For $x < 0$, $f(x) = -3$; for $0 < x < 1$, $f(x) = -1$

For $1 < x < 3$, $f(x) = 1 + 1 - 1 = 1$; for $x > 3$, $f(x) = 3$

Therefore, the solution set is $(1, 3)$

20. **D.** $\frac{11}{2}$

Explanation: $f(-x) = \frac{1}{1+e^{-x}} = \frac{e^x}{1+e^x}$, so $f(x) + f(-x) = 1$

In the sum, $\pm 1, \pm 2, \ldots, \pm 5$ form five pairs, each with sum 1

Also $f(0) = \frac{1}{2}$

Therefore, the required sum is $5 + \frac{1}{2} = \frac{11}{2}$

21. **A.** 3

Explanation: $f^{-1}(x) = \sqrt[3]{x}$; at an intersection point, $x^3 = \sqrt[3]{x}$

Cubing both sides, $x^9 = x \Rightarrow x(x^8 - 1) = 0$

For real x, $x = 0, 1, -1$; all three values satisfy the original equation

The intersection points are $(-1, -1), (0, 0), (1, 1)$; therefore, the number is 3

22. **B.** $0 < t \leq 2$

Explanation: On the branch $x < 0$, $x^2 = t$ gives only $x = -\sqrt{t}$, which exists exactly when $t > 0$
On the branch $x \geq 0$, $2 - x = t \Rightarrow x = 2 - t$, which is acceptable exactly when $t \leq 2$
The two branches have disjoint domains, so two distinct roots arise exactly when both branches have a root
Therefore, the necessary and sufficient condition is $0 < t \leq 2$

23. **C.** 168
Explanation: The first digit can be $3, 4, 5$; the last digit can be $1, 3, 5$ and cannot equal the first digit
For first digit $3, 4, 5$ the choices for the last digit are $2, 3, 2$ respectively; so there are 7 ordered end pairs
For each pair, the remaining four digits fill the middle four places in 4! ways
Therefore, the required number is $7 \times 4! = 168$

24. **A.** 3999960
Explanation: In any one position, a particular digit appears $4! = 24$ times
The sum of the digits is $1 + 2 + 3 + 4 + 5 = 15$; so in each position the digits contribute $24 \times 15 = 360$ in total
The sum of the place values is $1 + 10 + 100 + 1000 + 10000 = 11111$
Therefore, the required sum is $360 \times 11111 = 3999960$

25. **C.** 6720
Explanation: The three positions for A, B, C can be chosen in ${}^8C_3$ ways
In the chosen positions there is only one way to place A, B, C from left to right
The remaining five persons can be seated in the other five places in 5! ways
Therefore, the total is ${}^8C_3 \times 5! = 56 \times 120 = 6720$

26. **A.** 69120
Explanation: Treating the three subjects as three blocks, the mathematics block must be at the left or right end: 2 choices
The order of the other two blocks can be fixed in 2! ways
Arrangements of books within the blocks: $5!, 4!, 3!$ respectively
Therefore, the total is $2 \times 2! \times 5! \times 4! \times 3! = 69120$

27. **D.** 360
Explanation: To remove rotation, fix the first specified bead; the position opposite to it is then fixed for the second bead
The remaining six beads can be placed in 6! ways
Flipping the necklace gives the same necklace in the reverse order; since all beads are distinct, no arrangement coincides with its own mirror image, so we divide by 2
Therefore, the number of necklaces is $\frac{6!}{2} = 360$

28. **C.** 52
Explanation: If both specified persons are on the committee, choose 2 of the other 4 men and 2 of the other 4 women: ${}^4C_2 \times {}^4C_2 = 36$
If neither is on the committee, choose 3 of the other 4 men and 3 of the other 4 women: ${}^4C_3 \times {}^4C_3 = 16$
The two cases are mutually exclusive
Therefore, the total number of committees is $36 + 16 = 52$

29. **C.** 29
Explanation: Let the numbers of coins be x, y, z; then $x + 2y + 5z = 20$ with non-negative integers x, y, z
For $z = 0, 1, 2, 3, 4$, $x + 2y$ equals $20, 15, 10, 5, 0$ respectively
The numbers of possible values of y in these cases are $11, 8, 6, 3, 1$; for each y, x is determined
Therefore, the total number of ways is $11 + 8 + 6 + 3 + 1 = 29$

30. **B.** 350
Explanation: With A at both ends, the middle seven places hold A, A, B, B, B, C, C; the number of ways is $\frac{7!}{2!\,3!\,2!} = 210$
With B at both ends, the number of ways is $\frac{7!}{4!\,1!\,2!} = 105$
With C at both ends, the number of ways is $\frac{7!}{4!\,3!} = 35$
Therefore, the total number of arrangements is $210 + 105 + 35 = 350$

31. **D.** 216

Explanation: If the first digit is 3, the other 3 can be placed in the remaining four places in 4 ways; the remaining three places are filled from $0, 1, 2$ in 3^3 ways

This case gives $4 \times 3^3 = 108$ numbers

If the first digit is not 3, it is 1 or 2: 2 choices; the two 3's can be placed among the last four places in ${}^4C_2$ ways; the other two places have 3^2 choices

Therefore, the total is $108 + 2 \times {}^4C_2 \times 3^2 = 216$

32. **B.** 32

Explanation: 1 must be present and 2 absent; so an odd number of elements must be taken from the remaining six elements $3, 4, \ldots, 8$

The number of selections is ${}^6C_1 + {}^6C_3 + {}^6C_5$

Therefore, the required number is $6 + 20 + 6 = 32$

33. **B.** 126

Explanation: Choosing four vertices, the two diagonals of the convex quadrilateral they form give one interior intersection point

Conversely, each interior intersection of two diagonals is associated with four distinct vertices

Since no three diagonals are concurrent, different choices of four vertices never give the same intersection point

Therefore, the number of intersection points is ${}^9C_4 = 126$

34. **A.** 1800

Explanation: Choose the four rows containing rooks in ${}^5C_4$ ways and the four columns in ${}^6C_4$ ways

Match the four chosen rows with the four columns one-to-one in 4! ways

Each matching fixes four squares; because the rooks are identical, there is no further factor

Therefore, the total is ${}^5C_4 \times {}^6C_4 \times 4! = 5 \times 15 \times 24 = 1800$

35. **A.** 6

Explanation: Each diagonal of a rectangle inscribed in a circle is a diameter of that circle

Joining opposite vertices of a regular octagon gives 4 distinct diameters

The four endpoints of any two diameters form a rectangle

Therefore, the number of rectangles is ${}^4C_2 = 6$

36. **D.** 42

Explanation: Taking the digits in increasing order, the valid triples are

$(1, 2, 9), (1, 3, 8), (1, 4, 7), (1, 5, 6)$

The other triples are $(2, 3, 7), (2, 4, 6), (3, 4, 5)$; 7 in all

All digits are non-zero and distinct, so each triple gives $3! = 6$ numbers

Therefore, the total is $7 \times 6 = 42$

37. **B.** 36

Explanation: If all three colours appear among four balls, the colour counts form some arrangement of $(2, 1, 1)$

The colour that appears twice can be chosen in 3 ways; each colour has at least two balls available

For a fixed colour appearing twice, the colour sequences of the four balls number $\frac{4!}{2!} = 12$

Therefore, the total number of sequences is $3 \times 12 = 36$

38. **A.** 4608

Explanation: $r^2 = r(r-1) + r$, so split the sum into two parts

$r(r-1)\,{}^8C_r = 8 \cdot 7\,{}^6C_{r-2}$ and $r\,{}^8C_r = 8\,{}^7C_{r-1}$

Summing over $r \geq 2$ in the first part and $r \geq 1$ in the second gives $8 \cdot 7 \cdot 2^6 + 8 \cdot 2^7$

Therefore, the required value is $3584 + 1024 = 4608$

39. **D.** $[1, 4]$

Explanation: For the inverse sine function to be defined, its argument must lie between -1 and 1

$-1 \leq \log_2 \frac{x}{2} \leq 1 \Rightarrow 2^{-1} \leq \frac{x}{2} \leq 2^1$

$\Rightarrow \frac{1}{2} \leq \frac{x}{2} \leq 2 \Rightarrow 1 \leq x \leq 4$

Therefore, the domain is $[1, 4]$

40. **A.** 2880

Explanation: The word has 5 vowels (E, U, A, I, O) and 3 consonants (Q, T, N); no letter repeats

Treating the 5 vowels as one block gives $3 + 1 = 4$ units, which can be arranged in $4! = 24$ ways

Also, the 5 vowels inside the block can be arranged among themselves in $5! = 120$ ways

Therefore, the total number of arrangements is $24 \times 120 = 2880$

41. **A.** $\left[\frac{1}{3}, 3\right]$

Explanation: Let $y = \frac{x^2-x+1}{x^2+x+1}$

Cross-multiplying and collecting terms, $(y-1)x^2 + (y+1)x + (y-1) = 0$

Since x is real, the discriminant $\Delta \geq 0$

$(y+1)^2 - 4(y-1)^2 \geq 0 \Rightarrow (3-y)(3y-1) \geq 0$

$\Rightarrow (y-3)(3y-1) \leq 0$

Therefore, $\frac{1}{3} \leq y \leq 3$

42. **B.** 639

Explanation: From 3 identical white balls, any number can be selected in $3 + 1 = 4$ ways

From 4 identical black balls, the number of ways is $4 + 1 = 5$

From 5 balls of different colours, the number of ways is $2^5 = 32$

The total number of ways is $4 \times 5 \times 32 = 640$; one of these selects no ball at all

Therefore, the number of ways to select at least one ball is $640 - 1 = 639$

43. **B.** $f(x) = x \cdot \frac{a^x-1}{a^x+1}$

Explanation: For $f(x) = x \cdot \frac{a^x-1}{a^x+1}$, $f(-x) = -x \cdot \frac{a^{-x}-1}{a^{-x}+1} = -x \cdot \frac{1-a^x}{1+a^x} = x \cdot \frac{a^x-1}{a^x+1} = f(x)$

Since $f(-x) = f(x)$, it is an even function

44. **D.** 60

Explanation: The number of paths from $(0, 0)$ to $(2, 3)$ is ${}^5C_2 = 10$

From $(2, 3)$ to $(4, 5)$ we need $4 - 2 = 2$ steps right and $5 - 3 = 2$ steps up

The number of paths for this part is ${}^4C_2 = 6$

Therefore, the total number of shortest paths through $(2, 3)$ is $10 \times 6 = 60$

45. **C.** $-\frac{29}{20}$

Explanation: Putting $x = 2$ and $x = \frac{1}{2}$ in turn:

$2f(2) + 3f\left(\frac{1}{2}\right) = 4$ ...(i)

$2f\left(\frac{1}{2}\right) + 3f(2) = \frac{1}{4}$ ...(ii)

Multiplying (i) by 2 and (ii) by 3: $4f(2) + 6f\left(\frac{1}{2}\right) = 8$ and $9f(2) + 6f\left(\frac{1}{2}\right) = \frac{3}{4}$

Subtracting, $5f(2) = \frac{3}{4} - 8 = -\frac{29}{4}$

Therefore, $f(2) = -\frac{29}{20}$

46. **A.** $(-\infty, -1] \cup [1, \infty)$

Explanation: The composite function is $(f \circ g)(x) = f(x^2 + 1) = \sqrt{(x^2+1)-2} = \sqrt{x^2-1}$

For it to be real and defined, the expression under the root must be non-negative

$x^2 - 1 \geq 0 \Rightarrow (x-1)(x+1) \geq 0$

The solution is $x \leq -1$ or $x \geq 1$; that is, the domain is $(-\infty, -1] \cup [1, \infty)$

47. **B.** 5775

Explanation: 12 players are to be split into 3 teams of 4

If the teams had different names, the number of ways would be $\frac{12!}{(4!)^3} = 34650$

Since the teams have no specific names or identities (they are identical), we divide by 3!

Therefore, the total number of ways is $\frac{34650}{3!} = \frac{34650}{6} = 5775$

48. **C.** $\log_2\left(x + \sqrt{x^2+1}\right)$

Explanation: Let $y = \frac{2^x-2^{-x}}{2} \Rightarrow 2y = 2^x - \frac{1}{2^x} \Rightarrow (2^x)^2 - 2y(2^x) - 1 = 0$

By the quadratic formula, $2^x = \frac{2y \pm \sqrt{4y^2+4}}{2} = y \pm \sqrt{y^2+1}$

Since $2^x > 0$ and $y - \sqrt{y^2+1} < 0$, only the positive value is acceptable

$2^x = y + \sqrt{y^2+1} \Rightarrow x = \log_2\left(y + \sqrt{y^2+1}\right)$

Therefore, $f^{-1}(x) = \log_2\left(x + \sqrt{x^2+1}\right)$

49. **A.** 28

Explanation: Let $x = a + 2$, $y = b + 3$ and $z = c + 4$, where $a, b, c \geq 0$

Substituting in the given equation: $(a + 2) + (b + 3) + (c + 4) = 15 \Rightarrow a + b + c = 6$

The number of non-negative integer solutions with n variables and sum r is ${}^{n+r-1}C_r$

Here $n = 3$, $r = 6$; therefore, the number of solutions is ${}^8C_6 = {}^8C_2 = 28$

50. **A.** $\frac{\pi}{2}$

Explanation: $f(x) = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x = 1 - \frac{1}{2}(2 \sin x \cos x)^2$

$= 1 - \frac{1}{2}\sin^2 2x = 1 - \frac{1}{4}(1 - \cos 4x)$

$= \frac{3}{4} + \frac{1}{4}\cos 4x$

The fundamental period of $\cos x$ is 2π, so that of $\cos 4x$ is $\frac{2\pi}{4} = \frac{\pi}{2}$

Therefore, the fundamental period of $f(x)$ is $\frac{\pi}{2}$