Engineering Daily 18 [Set 04] EV - Answer Sheet

Trigonometric Ratios + Trigonometric Ratios of Associated Angles + Inverse Trigonometric Functions and Trigonometric Equations

Total Marks: 50 **Negative Marks:** 0.25 **Time:** 1 Hour

MCQ 50 marks

1. **A.** $\frac{\pi}{4}$

Explanation: $f^2(x) = \sin^2 2x + \cos^2 2x + 2|\sin 2x \cos 2x| = 1 + |\sin 4x|$

Since $f(x) > 0$, $f(x) = \sqrt{1 + |\sin 4x|}$

The period of $|\sin t|$ is π, so the fundamental period of $|\sin 4x|$ is $\frac{\pi}{4}$

f and f^2 have the same period

Therefore, the fundamental period is $\frac{\pi}{4}$

2. **A.** $-\frac{336}{625}$

Explanation: From $(\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = 1$, $\sec\theta - \tan\theta = \frac{1}{3}$

Adding and subtracting, $\sec\theta = \frac{5}{3}$ and $\tan\theta = \frac{4}{3}$

So, $\cos\theta = \frac{3}{5}$ and $\sin\theta = \frac{4}{5}$

$\sin 2\theta = \frac{24}{25}$ and $\cos 2\theta = \frac{9-16}{25} = -\frac{7}{25}$

$\sin 4\theta = 2\sin 2\theta \cos 2\theta = 2 \times \frac{24}{25} \times \left(-\frac{7}{25}\right)$

Therefore, the value is $-\frac{336}{625}$

3. **B.** $\left[5 - 2\sqrt{2}, 5 + 2\sqrt{2}\right]$

Explanation: $f(x) = \frac{3}{2}(1 - \cos 2x) + \frac{7}{2}(1 + \cos 2x) + 2\sin 2x$

$f(x) = 5 + 2\cos 2x + 2\sin 2x = 5 + 2\sqrt{2}\sin\left(2x + \frac{\pi}{4}\right)$

Since $-1 \le \sin\left(2x + \frac{\pi}{4}\right) \le 1$, $5 - 2\sqrt{2} \le f(x) \le 5 + 2\sqrt{2}$

Therefore, the range is $\left[5 - 2\sqrt{2}, 5 + 2\sqrt{2}\right]$

4. **B.** $\left[\frac{3-\sqrt{3}}{4}, \frac{3+\sqrt{3}}{4}\right]$

Explanation: Let $y = \frac{2+\sin x}{3+\cos x}$ (since $3 + \cos x \ge 2 > 0$)

$\sin x - y\cos x = 3y - 2$

The range of the left side is $\left[-\sqrt{1+y^2}, \sqrt{1+y^2}\right]$, so $|3y - 2| \le \sqrt{1+y^2}$

$(3y-2)^2 \le 1 + y^2$

$8y^2 - 12y + 3 \le 0$

So, $\frac{3-\sqrt{3}}{4} \le y \le \frac{3+\sqrt{3}}{4}$

Therefore, the range is $\left[\frac{3-\sqrt{3}}{4}, \frac{3+\sqrt{3}}{4}\right]$

5. **D.** 49

Explanation: $E = 9\csc^2 x + 16\sec^2 x = 9\left(1 + \cot^2 x\right) + 16\left(1 + \tan^2 x\right)$

$E = 25 + 9\cot^2 x + 16\tan^2 x$

By A.M. $\ge$ G.M., $9\cot^2 x + 16\tan^2 x \ge 2\sqrt{144\cot^2 x \tan^2 x}$

$= 2 \times 12 = 24$

So, $E \ge 25 + 24 = 49$

Therefore, the minimum value is 49

6. **C.** $\left[-1, \frac{1}{2} + \sqrt{2}\right]$

Explanation: Let $t = \sin x + \cos x$; then $-\sqrt{2} \le t \le \sqrt{2}$ and $\sin x \cos x = \frac{t^2-1}{2}$

$f = t + \frac{t^2-1}{2} = \frac{(t+1)^2}{2} - 1$

$t = -1$ lies in the interval, so the minimum value is -1

For $t = \sqrt{2}$, $f = \frac{1}{2} + \sqrt{2}$ and for $t = -\sqrt{2}$, $f = \frac{1}{2} - \sqrt{2}$; so the maximum value is $\frac{1}{2} + \sqrt{2}$

Therefore, the range is $\left[-1, \frac{1}{2} + \sqrt{2}\right]$

7. **B.** 2702

Explanation: $(\sin\theta + \cos\theta)^2 = \frac{3}{2}$, so $\sin\theta\cos\theta = \frac{1}{4}$

$\tan^2\theta + \cot^2\theta = \frac{\sin^4\theta + \cos^4\theta}{\sin^2\theta\cos^2\theta} = \frac{1 - 2\times\frac{1}{16}}{\frac{1}{16}} = 14$

Let $u = \tan^2\theta$, $v = \cot^2\theta$; then $u + v = 14$ and $uv = 1$

$u^3 + v^3 = (u+v)^3 - 3uv(u+v) = 14^3 - 3 \times 14$

Therefore, the value is $2744 - 42 = 2702$

8. **A.** $-\frac{7}{8}$

Explanation: Squaring and adding the two equations, $(\sin x + \sin y)^2 + (\cos x + \cos y)^2 = \frac{9}{4} + \frac{1}{4} = \frac{5}{2}$

$2 + 2(\sin x \sin y + \cos x \cos y) = \frac{5}{2}$

So, $\cos(x - y) = \frac{1}{4}$

$\cos 2(x - y) = 2\cos^2(x - y) - 1 = 2 \times \frac{1}{16} - 1$

Therefore, the value is $-\frac{7}{8}$

9. **A.** $\frac{3}{16}$

Explanation: We know, $\cos\theta\cos(60° - \theta)\cos(60° + \theta) = \frac{1}{4}\cos 3\theta$

Putting $\theta = 10°$, $\cos 10° \cos 50° \cos 70° = \frac{1}{4}\cos 30° = \frac{\sqrt{3}}{8}$

So, the given expression = $\frac{\sqrt{3}}{8} \times \cos 30° = \frac{\sqrt{3}}{8} \times \frac{\sqrt{3}}{2}$

Therefore, the value is $\frac{3}{16}$

10. **C.** $\frac{57}{64}$

Explanation: $\sin 50° = \sin 130°$, so we may take $\theta = 10°, 70°, 130°$

We know, $\sin^6\theta = \frac{10-15\cos 2\theta+6\cos 4\theta-\cos 6\theta}{32}$

Here, $\cos 20° + \cos 140° + \cos 260° = 0$ and $\cos 40° + \cos 280° + \cos 520° = 0$

Again, for each of the three angles $\cos 6\theta = \cos 60° = \frac{1}{2}$

$\frac{30-3\times\frac{1}{2}}{32} = \frac{57}{64}$

Therefore, the value is $\frac{57}{64}$

11. **B.** 9

Explanation: For $\theta = 10°, 70°, 130°$, $3\theta = 30°, 210°, 390°$, so $\tan 3\theta = \frac{1}{\sqrt{3}}$

Putting $t = \tan\theta$, $\frac{3t-t^3}{1-3t^2} = \frac{1}{\sqrt{3}}$

$t^3 - \sqrt{3}t^2 - 3t + \frac{1}{\sqrt{3}} = 0$

Its three distinct roots are $u = \tan 10°$, $v = \tan 70°$, $w = \tan 130°$

By the relations between roots and coefficients, $u + v + w = \sqrt{3}$ and $uv + vw + wu = -3$

$u^2 + v^2 + w^2 = (u + v + w)^2 - 2(uv + vw + wu) = 3 + 6$

Therefore, the value is 9

12. **D.** $\cot\frac{\pi}{96} - \frac{32}{\sqrt{3}}$

Explanation: We know, $\cot u - 2\cot 2u = \tan u$

$2^k \tan\left(2^k x\right) = 2^k \cot\left(2^k x\right) - 2^{k+1}\cot\left(2^{k+1}x\right)$

Summing $k = 0$ to 4, the middle terms cancel, so the sum = $\cot x - 32\cot 32x$

For $x = \frac{\pi}{96}$, $32x = \frac{\pi}{3}$ and $\cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$

Therefore, the value is $\cot\frac{\pi}{96} - \frac{32}{\sqrt{3}}$

13. **A.** $\frac{5}{4}$

Explanation: Let $\theta = \frac{\pi}{7}$ and $S = \cos 2\theta + \cos 4\theta + \cos 6\theta$

$2\sin\theta\, S = (\sin 3\theta - \sin\theta) + (\sin 5\theta - \sin 3\theta) + (\sin 7\theta - \sin 5\theta)$

$= \sin 7\theta - \sin\theta = -\sin\theta$, so $S = -\frac{1}{2}$

So, the given expression = $\frac{3}{2} + \frac{1}{2}S$

$= \frac{3}{2} - \frac{1}{4}$

Therefore, the value is $\frac{5}{4}$

14. **B.** $\frac{\pi}{3}$

Explanation: The range of $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sin\frac{2\pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right) = \sin\frac{\pi}{3}$

Since $\frac{\pi}{3}$ lies in the range

Therefore, the principal value is $\frac{\pi}{3}$

15. **D.** $\frac{16}{65}$

Explanation: α is in the second quadrant, so $\cos\alpha = -\sqrt{1 - \frac{9}{25}} = -\frac{4}{5}$

β is in the fourth quadrant, so $\sin\beta = -\sqrt{1 - \frac{144}{169}} = -\frac{5}{13}$

$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= \frac{3}{5} \times \frac{12}{13} - \left(-\frac{4}{5}\right)\left(-\frac{5}{13}\right) = \frac{36-20}{65}$

Therefore, the value is $\frac{16}{65}$

16. **A.** 25 cm^2 **and** 2 **radians**

Explanation: Let the radius be r cm and the arc length be l cm

Given, $2r + l = 20$, i.e. $l = 20 - 2r$

$S = \frac{1}{2}rl = \frac{1}{2}r(20 - 2r) = 10r - r^2$

$S = 25 - (r - 5)^2 \leq 25$; equality holds when $r = 5$

Then $l = 20 - 2 \times 5 = 10$ and $\theta = \frac{l}{r} = \frac{10}{5} = 2$ radians

Therefore, the maximum area is 25 cm^2 and the central angle is 2 radians

17. **C.** $\frac{1+\sqrt{10}}{3}$

Explanation: $\tan 2\alpha = \frac{\tan(\alpha+\beta)+\tan(\alpha-\beta)}{1-\tan(\alpha+\beta)\tan(\alpha-\beta)} = \frac{7+\frac{1}{2}}{1-\frac{7}{2}} = -3$

Taking $t = \tan\alpha$, $\frac{2t}{1-t^2} = -3$

$3t^2 - 2t - 3 = 0$

So, $t = \frac{1\pm\sqrt{10}}{3}$

Since α is acute, $t > 0$

Therefore, $\tan\alpha = \frac{1+\sqrt{10}}{3}$

18. **A.** 60°

Explanation: From the sine rule, $\sin A = \frac{a}{2R}$, $\sin B = \frac{b}{2R}$, $\sin C = \frac{c}{2R}$

Substituting in the condition and multiplying by $4R^2$, $a^2 + b^2 - c^2 = ab$

By the cosine rule, $\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{1}{2}$

Since $0 < C < \pi$, $C = 60°$
Therefore, $\angle C = 60°$

19. **B.** $-\frac{47}{49}$

Explanation: By the sine rule, $a : b : c = 5 : 7 : 8$; let $a = 5k$, $b = 7k$, $c = 8k$
$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{25+49-64}{2\times5\times7} = \frac{1}{7}$
$\cos 2C = 2\cos^2 C - 1 = \frac{2}{49} - 1 = -\frac{47}{49}$
Therefore, $\cos 2C = -\frac{47}{49}$

20. **D.** 2

Explanation: Since $A + B = \pi - C$, $\cot(A + B) = -\cot C$
$\frac{\cot A \cot B - 1}{\cot A + \cot B} = -\cot C$
So, $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$
$\cot^2 A + \cot^2 B + \cot^2 C = (\cot A + \cot B + \cot C)^2 - 2(\cot A \cot B + \cot B \cot C + \cot C \cot A) = 2^2 - 2 = 2$
Therefore, the required value is 2

21. **D.** $\frac{7}{8}$

Explanation: By the known triangle identity, $\cos A + \cos B + \cos C = 1 + \frac{r}{R}$
Here $\frac{r}{R} = \frac{1}{4}$; so $\cos A + \cos B + \cos C = \frac{5}{4}$
$\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} = \frac{3-(\cos A+\cos B+\cos C)}{2}$
$= \frac{3-\frac{5}{4}}{2} = \frac{7}{8}$
Therefore, the required value is $\frac{7}{8}$

22. **C.** $\sqrt{2} - 1$

Explanation: We know, $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$
So, $2\tan^{-1} x + 2\left(\tan^{-1} x + \cot^{-1} x\right) = \frac{5\pi}{4}$
or, $2\tan^{-1} x + \pi = \frac{5\pi}{4}$
or, $2\tan^{-1} x = \frac{\pi}{4}$
or, $\tan^{-1} x = \frac{\pi}{8}$
Therefore, $x = \tan\frac{\pi}{8} = \sqrt{2} - 1$

23. **C.** $\frac{323}{325}$

Explanation: $\cos A = \frac{14^2+15^2-13^2}{2\times14\times15} = \frac{3}{5}$
$\cos B = \frac{13^2+15^2-14^2}{2\times13\times15} = \frac{33}{65}$
Since these are angles of a triangle, $\sin A = \frac{4}{5}$
and $\sin B = \sqrt{1 - \frac{1089}{4225}} = \frac{56}{65}$
$\cos(A - B) = \cos A \cos B + \sin A \sin B = \frac{3}{5} \times \frac{33}{65} + \frac{4}{5} \times \frac{56}{65} = \frac{99+224}{325} = \frac{323}{325}$
Therefore, $\cos(A - B) = \frac{323}{325}$

24. **B.** $\frac{17}{108}$

Explanation: From the half-angle formulae, $\tan\frac{A}{2}\tan\frac{B}{2} = \frac{s-c}{s} = \frac{1}{5}$
$5s - 5c = s \Rightarrow 4s = 5c \Rightarrow 2(a + b) = 3c$
Taking $a = 2k$, $b = 3k$, we get $c = \frac{10k}{3}$
$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{4+9-\frac{100}{9}}{12} = \frac{17}{108}$
Therefore, $\cos C = \frac{17}{108}$

25. **D.** $\frac{1}{4}$

Explanation: From $A + B + C = \pi$ and $A = 2B$, $C = \pi - 3B$
$\frac{c}{b} = \frac{\sin C}{\sin B} = \frac{\sin 3B}{\sin B} = 3 - 4\sin^2 B = \frac{3}{2}$
So, $\sin^2 B = \frac{3}{8}$
$\cos A = \cos 2B = 1 - 2\sin^2 B = 1 - \frac{3}{4} = \frac{1}{4}$
Therefore, $\cos A = \frac{1}{4}$

26. **C.** 60°

Explanation: Let $u = \frac{A+B}{2}$; then $\frac{A-B}{2} = 30°$ and $0 < u < 90°$
$\sin A + \sin B = 2\sin u \cos 30° = \sqrt{3}\sin u$
$\sin C = \sin(A + B) = 2\sin u \cos u$
Substituting in the condition, $\sqrt{3}\sin u = 2\sqrt{3}\sin u \cos u$
Since $\sin u > 0$, $\cos u = \frac{1}{2}$, i.e. $u = 60°$
$C = 180° - 2u = 60°$
Therefore, $C = 60°$

27. **A.** 3 : 5 : 7

Explanation: Let $a = b - d$ and $c = b + d$. By the cosine rule, $c^2 = a^2 + b^2 - 2ab\cos 120° = a^2 + b^2 + ab$
$(b + d)^2 = (b - d)^2 + b^2 + b(b - d)$
$b^2 + 2bd + d^2 = 3b^2 - 3bd + d^2$
So, $5bd = 2b^2$, i.e. $d = \frac{2b}{5}$ $(b > 0)$
$a : b : c = \frac{3b}{5} : b : \frac{7b}{5}$
Therefore, $a : b : c = 3 : 5 : 7$

28. **D.** $\frac{49}{36}$

Explanation: Let $u = \tan\frac{A}{2}$, $v = \tan\frac{B}{2}$, $w = \tan\frac{C}{2}$
As $\frac{A+B+C}{2} = \frac{\pi}{2}$, $uv + vw + wu = 1$
Known result: $u + v + w = \frac{4R+r}{s}$
$u + v + w = \frac{20+2}{12} = \frac{11}{6}$
$u^2 + v^2 + w^2 = (u + v + w)^2 - 2(uv + vw + wu) = \frac{121}{36} - 2$
Therefore, the value is $\frac{49}{36}$

29. **A.** $\frac{5\pi}{12}$

Explanation: The principal range of $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and that of $\cos^{-1}$ is $[0, \pi]$
$\sin\frac{17\pi}{12} = -\sin\frac{5\pi}{12}$, so $\sin^{-1}\left(\sin\frac{17\pi}{12}\right) = -\frac{5\pi}{12}$
$\cos\frac{7\pi}{6} = \cos\frac{5\pi}{6}$, so $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \frac{5\pi}{6}$
$-\frac{5\pi}{12} + \frac{5\pi}{6}$
Therefore, the value is $\frac{5\pi}{12}$

30. **C.** $\pi + \tan^{-1}\frac{2}{3}$
Explanation: Let $u = \tan^{-1}2 + \tan^{-1}3$; then $\tan u = \frac{2+3}{1-6} = -1$
Both angles are greater than $\frac{\pi}{4}$ and less than $\frac{\pi}{2}$, so $\frac{\pi}{2} < u < \pi$, i.e. $u = \frac{3\pi}{4}$
Let $S = \frac{3\pi}{4} + \tan^{-1}5$; then $\tan S = \frac{-1+5}{1+5} = \frac{2}{3}$
$\frac{3\pi}{4} < S < \frac{5\pi}{4}$ and $\tan S > 0$, so $\pi < S < \frac{5\pi}{4}$
Therefore, $S = \pi + \tan^{-1}\frac{2}{3}$

31. **D.** $\pi - 2\tan^{-1}x$
Explanation: Let $\theta = \tan^{-1}x$; since $x > 1$, $\frac{\pi}{4} < \theta < \frac{\pi}{2}$
$\frac{2x}{1+x^2} = \sin 2\theta$ and $\frac{\pi}{2} < 2\theta < \pi$
$\sin 2\theta = \sin(\pi - 2\theta)$, where $0 < \pi - 2\theta < \frac{\pi}{2}$
Therefore, $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \pi - 2\tan^{-1}x$

32. **B.** $(-\infty, -1] \cup [1, \infty)$
Explanation: For real values, $-1 \le \frac{x^2-3}{x^2+1} \le 1$; here $x^2 + 1 > 0$
The right inequality $x^2 - 3 \le x^2 + 1$ is always true
From the left inequality, $-(x^2 + 1) \le x^2 - 3$, or $2x^2 \ge 2$, i.e. $|x| \ge 1$
Therefore, the domain is $(-\infty, -1] \cup [1, \infty)$

33. **B.** $\frac{1}{2\sqrt{5+2\sqrt{3}}}$
Explanation: The domain is $|x| \le \frac{1}{2}$; since the sum is positive, $x > 0$
From $\sin^{-1}(2x) = \frac{\pi}{6} - \sin^{-1}x$, $2x = \frac{1}{2}\sqrt{1-x^2} - \frac{\sqrt{3}}{2}x$
$(4 + \sqrt{3})x = \sqrt{1-x^2}$
Squaring, $(19 + 8\sqrt{3})x^2 = 1 - x^2$
$x^2 = \frac{1}{20+8\sqrt{3}} = \frac{1}{4(5+2\sqrt{3})}$
Therefore, $x = \frac{1}{2\sqrt{5+2\sqrt{3}}}$

34. **A.** 1
Explanation: Taking tangent on both sides, $\frac{2x}{5-x^2} = \frac{1}{2}$
$x^2 + 4x - 5 = 0$
$(x - 1)(x + 5) = 0$
So, $x = 1$ or $x = -5$
For $x = -5$ the left side is negative while the right side is positive; so it is rejected
For $x = 1$ the left side is $\tan^{-1}3 - \frac{\pi}{4}$, which lies in $\left(0, \frac{\pi}{4}\right)$ and has tangent $\frac{1}{2}$
Therefore, the only solution is $x = 1$

35. **B.** $\tan^{-1}\frac{6}{13}$
Explanation: $\frac{3}{9n^2+3n-1} = \frac{(3n+2)-(3n-1)}{1+(3n+2)(3n-1)}$
For $n \ge 1$, $0 < \tan^{-1}(3n + 2) - \tan^{-1}(3n - 1) < \frac{\pi}{2}$, so each term $= \tan^{-1}(3n + 2) - \tan^{-1}(3n - 1)$
Adding, the middle terms cancel and $S = \tan^{-1}32 - \tan^{-1}2$
$S = \tan^{-1}\frac{32-2}{1+32\times2} = \tan^{-1}\frac{30}{65}$
Therefore, the value is $\tan^{-1}\frac{6}{13}$

36. **B.** $\frac{\sqrt{14}}{4}$
Explanation: Let $u = \tan^{-1}x$, $v = \tan^{-1}(2x)$, $w = \tan^{-1}(4x)$; all three lie in $\left(0, \frac{\pi}{2}\right)$
If $u + v + w = \pi$, then $\tan u + \tan v + \tan w = \tan u \tan v \tan w$
$x + 2x + 4x = x \times 2x \times 4x$
$7x = 8x^3$
So, $x^2 = \frac{7}{8}$ and as $x > 0$, $x = \frac{\sqrt{14}}{4}$
For this value $\sin(u + v + w) = 0$ and $0 < u + v + w < \frac{3\pi}{2}$, so the sum is π
Therefore, $x = \frac{\sqrt{14}}{4}$

37. **C.** $2\tan^{-1}2$
Explanation: Here, $x \times \frac{4}{x} = 4 > 1$, so the sum lies in $\left(\frac{\pi}{2}, \pi\right)$
$\tan^{-1}x + \tan^{-1}\frac{4}{x} = \pi + \tan^{-1}\frac{x+\frac{4}{x}}{1-4} = \pi - \tan^{-1}\frac{x+\frac{4}{x}}{3}$
By A.M. $\ge$ G.M., $x + \frac{4}{x} \ge 4$, with equality at $x = 2$
So, the value is $\le \pi - \tan^{-1}\frac{4}{3}$, and at $x = 2$ the value is $2\tan^{-1}2$
Again, $\tan\left(2\tan^{-1}2\right) = \frac{4}{1-4} = -\frac{4}{3}$, so $2\tan^{-1}2 = \pi - \tan^{-1}\frac{4}{3}$
Therefore, the maximum value is $2\tan^{-1}2$

38. **A.** $x^2 + y^2 + xy = \frac{3}{4}$
Explanation: Let $u = \sin^{-1}x$; then $y = \sin\left(\frac{\pi}{3} - u\right)$

$y = \frac{\sqrt{3}}{2}\sqrt{1-x^2} - \frac{x}{2}$, since $\cos u \geq 0$
$2y + x = \sqrt{3}\sqrt{1-x^2}$
Squaring, $4y^2 + 4xy + x^2 = 3 - 3x^2$
Therefore, $x^2 + y^2 + xy = \frac{3}{4}$

39. **D.** 8
Explanation: Domain conditions: $\cos 3x \not\equiv 0$ and $\cos 2x \not\equiv 0$
From the equation, $\sin 3x \sin 2x = \cos 3x \cos 2x$, i.e. $\cos 5x = 0$
So, $x = \frac{(2k+1)\pi}{10}$; for $k = 0, 1, \ldots, 9$ these give 10 candidate values in $[0, 2\pi)$
Among them, $\cos 3x = 0$ at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$, so these are excluded
For no other value is $\cos 3x$ or $\cos 2x$ zero
Therefore, there are $10 - 2 = 8$ solutions

40. **C.** $\left(1, \frac{5}{4}\right)$
Explanation: Taking $t = \cos x \in [-1, 1]$, $k = 1 - t - t^2 = \frac{5}{4} - \left(t + \frac{1}{2}\right)^2$
Each $t \in (-1, 1)$ gives 2 values of x, but $t = \pm 1$ gives 1 each
So for exactly 4 solutions both distinct values of t must lie in $(-1, 1)$
At $t = -1$, $k = 1$, and the maximum of the parabola is $\frac{5}{4}$ (at $t = -\frac{1}{2}$)
For $k = 1$, $t = 0, -1$ give 3 solutions, and for $k = \frac{5}{4}$, $t = -\frac{1}{2}$ gives 2 solutions
Therefore, $1 < k < \frac{5}{4}$

41. **A.** 6
Explanation: $\cos 4x = 1 - 2\sin^2 2x$ and $6 \sin x \cos x = 3 \sin 2x$
Taking $t = \sin 2x$, $1 - 2t^2 + 3t = 2$
$2t^2 - 3t + 1 = 0$
$(2t - 1)(t - 1) = 0$
So, $t = \frac{1}{2}$ or $t = 1$
In $0 \leq 2x < 4\pi$, $\sin 2x = \frac{1}{2}$ has 4 solutions and $\sin 2x = 1$ has 2; the two sets are disjoint
Therefore, there are $4 + 2 = 6$ solutions in total

42. **C.** 5
Explanation: From $\sin 3x = \sin\left(\frac{\pi}{2} - 2x\right)$ we get two cases
First, $3x = \frac{\pi}{2} - 2x + 2n\pi$, i.e. $x = \frac{\pi}{10} + \frac{2n\pi}{5}$; $n = 0, 1, 2, 3, 4$ give 5 solutions
Second, $3x = \pi - \left(\frac{\pi}{2} - 2x\right) + 2n\pi$, i.e. $x = \frac{\pi}{2} + 2n\pi$; only $x = \frac{\pi}{2}$ lies in the interval
$x = \frac{\pi}{2}$ is already in the first list for $n = 1$
Therefore, there are 5 distinct solutions

43. **D.** 8
Explanation: Both sides are non-negative, so squaring gives an equivalent equation
$1 + 2|\sin x \cos x| = \frac{3}{2}$
$|\sin 2x| = \frac{1}{2}$
So, $\sin 2x = \frac{1}{2}$ or $\sin 2x = -\frac{1}{2}$
In $0 \leq 2x < 4\pi$, each equation has 4 solutions
Therefore, there are $4 + 4 = 8$ solutions in total

44. **B.** 8
Explanation: Let $u = 4^{\sin^2 x}$; then $1 \leq u \leq 4$ and $4^{\cos^2 x} = \frac{4}{u}$
Given, $u + \frac{4}{u} = \frac{9}{2}$
$2u^2 - 9u + 8 = 0$
So, $u = \frac{9 \pm \sqrt{17}}{4}$; both roots lie in $(1, 4)$ and are distinct
So $\sin^2 x$ has two distinct values, each in $(0, 1)$
Each value gives 4 solutions in $[0, 2\pi)$
Therefore, there are $2 \times 4 = 8$ solutions in total

45. **B.** 6
Explanation: Using $\sin x + \sin 3x = 2 \sin 2x \cos x$, $\sin 2x(1 + 2\cos x) = 0$
$\sin 2x = 0$ gives $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$
$1 + 2\cos x = 0$ gives $\cos x = -\frac{1}{2}$, i.e. $x = \frac{2\pi}{3}, \frac{4\pi}{3}$
The two lists have no common value
Therefore, there are $4 + 2 = 6$ distinct solutions

46. **C.** $\frac{3\pi}{2}$
Explanation: $2\left(1 - \sin^2 x\right) + 3\sin x = 3$
$2\sin^2 x - 3\sin x + 1 = 0$
$(2\sin x - 1)(\sin x - 1) = 0$
$\sin x = \frac{1}{2}$ gives $x = \frac{\pi}{6}, \frac{5\pi}{6}$ and $\sin x = 1$ gives $x = \frac{\pi}{2}$
$\frac{\pi}{6} + \frac{5\pi}{6} + \frac{\pi}{2}$
Therefore, the sum is $\frac{3\pi}{2}$

47. **C.** $0, \frac{\pi}{2}$
Explanation: Dividing both sides by $\sqrt{2}$, $\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta = \frac{1}{\sqrt{2}}$
$\cos\left(\theta - \frac{\pi}{4}\right) = \cos\frac{\pi}{4}$
$\theta - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}$
For $0 \leq \theta \leq \pi$, taking $n = 0$, $\theta = \frac{\pi}{2}$ or $\theta = 0$
Therefore, the solutions are $0, \frac{\pi}{2}$

48. **D.** $\theta = n\pi \pm \frac{\pi}{6}$

Explanation: $3\tan^2\theta = 1$

$\tan^2\theta = \frac{1}{3} = \tan^2\frac{\pi}{6}$

If $\tan^2\theta = \tan^2\alpha$, the general solution is $\theta = n\pi \pm \alpha$

Therefore, $\theta = n\pi \pm \frac{\pi}{6}$

49. **A.** $\frac{1621}{4096}$

Explanation: Given, $(\sin\theta + \cos\theta)^2 = \frac{1}{4}$

or, $1 + 2\sin\theta\cos\theta = \frac{1}{4}$

So, $\sin\theta\cos\theta = -\frac{3}{8}$

Again, $(\sin\theta - \cos\theta)^2 = \frac{1}{4} - 4 \times \left(-\frac{3}{8}\right) = \frac{7}{4}$

or, $\sin\theta - \cos\theta = \frac{\sqrt{7}}{2}$

Adding and subtracting, $\sin\theta = \frac{1+\sqrt{7}}{4}$ and $\cos\theta = \frac{1-\sqrt{7}}{4}$

Putting these in the calculator, $\sin^{10}\theta + \cos^{10}\theta \approx 0.3957 = \frac{1621}{4096}$

Therefore, the value is $\frac{1621}{4096}$

50. **B.** $\frac{k-1}{k+1}\sin(\alpha+\beta)$

Explanation: Given, $\tan\alpha = k\tan\beta$

or, $\frac{\sin\alpha}{\cos\alpha} = k \times \frac{\sin\beta}{\cos\beta}$

or, $\frac{\sin\alpha\cos\beta}{\cos\alpha\sin\beta} = \frac{k}{1}$

By componendo and dividendo,

$\frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\sin\alpha\cos\beta + \cos\alpha\sin\beta} = \frac{k-1}{k+1}$

or, $\frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)} = \frac{k-1}{k+1}$

Therefore, $\sin(\alpha-\beta) = \frac{k-1}{k+1}\sin(\alpha+\beta)$